Løsningsforslag SØK 3004 Konte vår 2013 (SL)

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a) $\int (3x^2 + 2e^{2x})\,dx = \underline{\underline{x^3 + e^{2x} + C}}$

b) $\int x e^x \,dx$

Sett $f(x) = x$, $f'(x) = 1$, $g'(x) = e^x$, $g(x) = e^x$.

$\int x e^x \,dx = x e^x - \int 1 \cdot e^x \,dx = \underline{\underline{x e^x - e^x + C}}$

c) $\int (2x+8) e^{(x+4)^2} \,dx$

Sett $u = (x+4)^2$, $du = 2(x+4)\,dx$

$\int e^u \,du = e^u + C$

Innsatt for u:

$\int (2x+8) e^{(x+4)^2} \,dx = \underline{\underline{e^{(x+4)^2} + C}}$

d) $\int_0^1 \frac{1}{\sqrt{x}} \,dx$. Ser at integranden ikke er definert for $x = 0$ og at $\lim_{x \to 0^+} \frac{1}{\sqrt{x}} = \infty$.

$\int_0^1 \frac{1}{\sqrt{x}} \,dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt{x}} \,dx = \lim_{t \to 0^+} \left[2\sqrt{x} + C \right]_t^1$

$= \lim_{t \to 0^+} \left(2\sqrt{1} - 2\sqrt{t} \right) = \underline{\underline{2}}$

(2)

Løser a) og b) samtidig:

$$\begin{array}{cccc}\\ \\ \end{array}\left[\begin{array}{cccc|c|c} & & & & a) & b) \\ 4 & 3 & 2 & 1 & 0 & 1 \\ 12 & 9 & 3 & 4 & 0 & 10 \\ -4 & -3 & -2 & 4 & 0 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c|c} 4 & 3 & 2 & 1 & 0 & 1 \\ 0 & 0 & -3 & 1 & 0 & 7 \\ 0 & 0 & 0 & 5 & 0 & 5 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c|c} 4 & 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c|c} 4 & 3 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c|c} 1 & 3/4 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

Kan velge X_2 fritt: $X_2 = t$

a) $\underline{\underline{X_1 = -3/4t \;, X_2 = t \;, X_3 = 0 \;, X_4 = 0}}$

b) $X_1 + 3/4t = 1 \Leftrightarrow X_1 = 1 - 3/4t \;, X_2 = t, \; X_3 = -2, \; X_4 = 1,$

eller

$$\underline{\underline{\bar{X} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -3/4 \\ 1 \\ 0 \\ 0 \end{bmatrix} t}}$$

c) F.eks sett $t=0$: $\bar{X} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 4 & 3 & 2 & 1 \\ 12 & 9 & 3 & 4 \\ -4 & -3 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4\cdot 1 + 3\cdot 0 + 2(-2) + 1\cdot 1 \\ 12\cdot 1 + 9\cdot 0 + 3(-2) + 4\cdot 1 \\ -4\cdot 1 - 3\cdot 0 - 2(-2) + 4\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 4 \end{bmatrix}$$

(3)

a) Nasjonen står overfor følgende maksimeringsproblem:

$$\max_{\bar{x}} \sum_{j=1}^{n} \left(\alpha_j x_j - \frac{1}{2} \beta_j x_j^2 \right)$$

s.t

$$\bar{1}^T \bar{x} \leq C$$

$$x_j \geq 0 \quad \forall j = 1, \ldots, n.$$

På standardform:

$$\max_{\bar{x}} \sum_{j=1}^{n} \left(\alpha_j x_j - \frac{1}{2} \beta_j x_j^2 \right)$$

s.t

$$\bar{1}^T \bar{x} \leq C$$

$$-x_j \leq 0 \quad \forall j = 1, \ldots, n.$$

Vi får Lagrangefunksjonen

$$\mathcal{L} = \sum_{j=1}^{n} \left(\alpha_j x_j - \frac{1}{2} \beta_j x_j^2 \right) - \lambda \left(\bar{1}^T \bar{x} - C \right) + \sum_{j=1}^{n} \mu_j x_j$$

b) KKT-(nødvendig) betingelse for optimum:

$$\mathcal{L}_{x_j} = \alpha_j - \beta_j x_j - \lambda + \mu_j = 0 \quad , j=1,\dots,n \qquad ⊛$$

med

$$\lambda \geq 0 \,,\ \lambda = 0 \text{ hvis } \bar{1}^T \bar{x} < C$$

$$\mu_j \geq 0 \,,\ \mu_j = 0 \text{ hvis } x_j > 0 \quad , j=1,\dots,n$$

Fra ⊛ får vi at $x_j = \frac{\alpha_j}{\beta_j} - \frac{\lambda - \mu_j}{\beta_j}$

At hele budsjettet ikke brukes betyr at

$$\bar{1}^T\bar{x} < C \Rightarrow \lambda = 0 \Rightarrow x_j = \frac{\alpha_j}{\beta_j} + \frac{\mu_j}{\beta_j}$$

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \frac{\alpha_j}{\beta_j} + \sum_{j=1}^{n} \frac{\mu_j}{\beta_j} = H + \underbrace{\sum_{j=1}^{n} \frac{\mu_j}{\beta_j}}_{\geq 0} < C$$

(Da må også $H<C$)

Da ser vi at når ~~H<C, blir~~ ikke hele budsjettet (C) blir brukt, må $H<C$.

c) Når alle får funding/finansiering, må $\mu_j = 0 \ \forall j=1,\dots,n$.
Da får vi at

$$x_j = \frac{\alpha_j}{\beta_j} - \frac{\lambda}{\beta_j} > 0 \Leftrightarrow \alpha_j > \lambda .$$

$$\underbrace{\sum_{j=1}^{n} \frac{\alpha_j}{\beta_j} - \lambda \sum_{j=1}^{n} \frac{1}{\beta_j}}_{\sum_{j=1}^{n} x_j} - C \leq 0 \Leftrightarrow H - \lambda K - C \leq 0 \Leftrightarrow$$

$$\lambda \geq \frac{H-C}{K}.$$ Siden $\alpha_j > \lambda$, har vi at

$$\alpha_j > \frac{H-C}{K}.$$

④

$$-\frac{u''}{u'}w = k \Leftrightarrow u''w + u'k = 0$$

Sett $v = u'$:

$$v'w + vk = 0 \Leftrightarrow \frac{v'}{v} = -k\frac{1}{w} \quad \left(\begin{array}{c}\text{Husk at} \\ v' = \frac{dv}{dw}\end{array}\right)$$

$$\int \frac{dv}{v} = \int -k\frac{1}{w}\,dw \Leftrightarrow \ln|v| = -k\ln|w| + C \Leftrightarrow$$

$$|v| = A|w|^{-k}, \quad A = e^C.$$

Vi har oppgitt at $w > 0$ og $u' = v > 0$:

$$v = Aw^{-k}, \text{ dvs}$$

$$\frac{du}{dw} = Aw^{-k}.$$

$\underline{k \neq 1}$: $\int du = \int Aw^{-k}\,dw \Leftrightarrow \underline{\underline{u = A\frac{1}{1-k}w^{1-k} + B}}$

$\underline{k = 1}$: $\int du = \int Aw^{-k}\,dw \Leftrightarrow \underline{\underline{u = A\ln w + B}}$

(Oppgaven er en konvertering av en av de to anbefalte oppgavene (6.1(5)) til øving 8 den 19/10-2012.)