IF400 – Financial Derivatives

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Notatark skrevet for emnet IF400 – Finansielle instrumenter. Notatarket her er omfattende og dekker anslagsvis 95% av pensumet. Kapittel 1 til 6 er gjennomgått i større detalj med flere utledninger enn det som er gitt i forelesninger og i læreboken. Fra kapittel 7 og utover har jeg heller forsøkt å komprimere stoffet til det aller viktigste. Ellers følger notasjonen i stor grad læreboken «Derivatives Markets» av McDonald.

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Chapter 1: Fixed Income

Bonds

Consider a bond with a coupon payment of c_i in t_i years where $i = 1, \dots, n$. If the bond is compounded k times per year, the bond price becomes

$$P = \sum_{i=1}^{n} \frac{c_i/k}{(1 + y/k)^{kt_i}} + \frac{F}{(1 + y/k)^{kn}}$$

The parameter y is the bond's ***yield to maturity*** or simply the yield. This is the interest rate on the bond that corresponds with the price P given the coupon payments c_i.

A common convention is to use continuous compounding. This occurs when $k \to \infty$. Then, the price becomes

$$P = \sum_{i=1}^{n} c_i e^{-yt_i} + Fe^{-yn}$$

We can use zero-coupon bonds as an alternative way to price coupon bearing bonds. Let $r^{(k)}(0,t)$ be the yield on a zero-coupon bond with lifetime $[0,t]$ that is compounded k times annually. The price of a zero-coupon bond with a face value of F is then

$$P(0,t) = \frac{F}{\left(1 + r^{(k)}(0,t)\right)^{kt}}$$

Suppose $F = 1$. Then the price formula can be written in terms of the zero-coupon rate

$$r^{(k)}(t) = k\left(P(0,t)^{-\frac{1}{kt}} - 1\right)$$

As we can see, the zero-coupon rates can be converted to different compounding frequencies. That is, it is possible to find $r^{(s)}(t)$ given $r^{(k)}(t)$. Under continuous compounding where $k \to \infty$, we get

$$r^{(\infty)}(t) = -\frac{1}{t}\ln\big(P(0,t)\big)$$

The sequence $\{r^{(k)}(t) | t \geq 0\}$ is called the ***yield curve***. It is typically upward-sloping.

Forward Rates

Consider a period $[t_1, t_2]$. The **forward rate** $f(t_1, t_2)$ is the rate that we can lock in today for borrowing and lending over the period $[t_1, t_2]$. Since this is also an interest rate, we must decide on a compounding frequency.

Assume now that we have continuous compounding. For each 1 borrowed at time t_1, the amount that must be paid back at time t_2 is

$$e^{f(t_1,t_2)(t_2-t_1)}$$

It is possible to derive an expression for the forward rate. We will consider the following strategy

- Purchase one t_1 maturity zero-coupon bond
- Short $\frac{P(0,t_1)}{P(0,t_2)}$ zero-coupon bonds with maturity t_2

Transaction	$t = 0$	$t = t_1$	$t = t_2$
Purchase t_1 bond	$-P(0, t_1)$	1	$-$
Short t_2 bond	$\frac{P(0,t_1)}{P(0,t_2)} \cdot P(0, t_2)$	0	$-\frac{P(0,t_1)}{P(0,t_2)} \cdot 1$
Total	0	1	$-\frac{P(0,t_1)}{P(0,t_2)}$

This is equivalent to borrowing 1 at t_1 with a repayment of $\frac{P(0,t_1)}{P(0,t_2)}$ at t_2. Therefore,

$$e^{f(t_1,t_2)(t_2-t_1)} = \frac{P(0,t_1)}{P(0,t_2)}$$

Solving for the forward rate

$$f(t_1, t_2) = \frac{\ln P(0, t_1) - \ln d(t_2)}{t_2 - t_1}$$

Since $\ln P(0,t) = -tr(0,t)$ under continuous compounding, we can rewrite the forward rate equation as

$$f(t_1, t_2) = \frac{t_2 r(0, t_2) - t_1 r(0, t_1)}{t_2 - t_1}$$

We can also derive this relationship for other compounding frequencies. In this case, the forward rate becomes

$$f(t_1, t_2) = k\left[\left(\frac{P(0, t_1)}{P(0, t_2)}\right)^{\frac{1}{k(t_2 - t_1)}} - 1\right]$$

The forward rate can be used to price zero-coupon bonds that are issued at a later point in time. Suppose we want to find the time 0 price for a zero-coupon bond that is issued at time t_1 with maturity t_2 with annual compounding $k = 1$. Then

$$P_0(t_1, t_2) = \frac{1}{\left(1 + f(t_1, t_2)\right)^{t_2 - t_1}}$$

Using the no-arbitrage relationship

$$\left(1 + r(0, t_1)\right)^{t_1} \cdot \left(1 + f(t_1, t_2)\right)^{t_2 - t_1} = \left(1 + r(0, t_2)\right)^{t_2}$$

We can write the forward rate expression as

$$\left(1 + f(t_1, t_2)\right)^{t_2 - t_1} = \frac{\left(1 + r(0, t_2)\right)^{t_2}}{\left(1 + r(0, t_1)\right)^{t_1}}$$

$$\left(1 + f(t_1, t_2)\right)^{t_2 - t_1} = \frac{P(0, t_1)}{P(0, t_2)}$$

Then the time 0 price can be written as

$$P_0(t_1, t_2) = \frac{P(0, t_2)}{P(0, t_1)}$$

Given the price of zero-coupon bonds it is possible to price coupon bonds. Let $B_t(t,T,c,n)$ be the time t price of a bond that issued at time t with maturity T. Let c be the coupon rate, that is, $C = cF$. Finally, let n denote the number of payments. We have the relationship

$$B_t(t,T,c,n) = \sum_{i=1}^{n} cP_t(t,t_i) + F \cdot P_t(t,T)$$

Solving for the coupon rate gives

$$c = \frac{B_t(t,T,c,n) - F \cdot P_t(t,T)}{\sum_{i=1}^{n} P_t(t,t_i)}$$

The **par coupon** is the coupon rate which ensures that the bond price is equal to the face value. Then, the par coupon can be found by the following

$$c = F\left(\frac{1 - P_t(t,T)}{\sum_{i=1}^{n} P_t(t,t_i)}\right)$$

A common convention is to assume that $F = 1$, then

$$c = \left(\frac{1 - P_t(t,T)}{\sum_{i=1}^{n} P_t(t,t_i)}\right)$$

Duration and Immunization

Bond prices are subject to interest rate risk. When the bond yield changes, so does the price of the bond. Suppose we have a bond that makes m coupon payments annually for T years with a face value of F. The per-period yield is then y/m with y as the annualized yield to maturity. Then, the number of periods until maturity is $n = m \cdot T$. The price of this bond becomes

$$B(y) = \sum_{i=1}^{T} \frac{C/m}{(1+y/m)^i} + \frac{F}{(1+y/k)^n}$$

The dollar change of the price when the yield increases by 1 is its partial derivative with respect to the yield

$$\frac{\Delta B}{\Delta y} = \frac{\partial B}{\partial y} = -\frac{1}{1+\frac{y}{m}}\left(\sum_{i=1}^{T} \frac{i}{m} \cdot \frac{C/m}{\left(1+\frac{y}{m}\right)^{i}} + \frac{n}{m} \cdot \frac{F}{\left(1+\frac{y}{m}\right)^{n}}\right)$$

To get the change in percentage points, we can divide this derivative by 100. Another method for measuring interest rate risk is using the **modified duration**, which gives the percentage change in the bond price for a unit change in the yield. This measure is more useful as it allows us to compare interest rate risk for different bonds. The modified duration can be expressed as

$$D^{MOD} = -\frac{1}{B}\left(\frac{\Delta B}{\Delta y}\right)$$

Since bond prices fall when the yield increases, we multiply the price change by -1 to get the percentage change in absolute terms. Another measure of bond price risk is the **Macauley duration**, which is given as

$$D^{MAC} = D^{MOD}\left(1+\frac{y}{m}\right)$$

For simplicity, suppose $m = 1$. Then,

$$D^{MAC} = -\frac{\Delta B}{B} \cdot \frac{1+y}{\Delta y}$$

$$D^{MAC} = \frac{1}{B(y)}\left(\sum_{i=1}^{T} \frac{i}{m} \cdot \frac{C/m}{\left(1+\frac{y}{m}\right)^{i}} + \frac{n}{m} \cdot \frac{F}{\left(1+\frac{y}{m}\right)^{n}}\right)$$

This measure has a meaningful interpretation. It is a weighted average of the time until the bond payments occur. Macaulay duration is useful because it can provide a way of measuring risk for bonds that differ in maturities and number of payments. A change in the yield will have a larger effect on bonds with long maturities because the change in the payments are greater ahead in time.

We can use a Taylor series approximation to derive the sensitivity of the bond price. Recall that the first order approximation of a function $f(x)$ is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Let $f(x_0) = B(y)$ and $x = y + \epsilon$ where ϵ is a small number. Then

$$B(y + \epsilon) \approx B(y) + B'(y)\epsilon$$

Then the change in price is

$$B(y + \epsilon) - B(y) \approx B'(y)\epsilon$$

$$\Delta B \approx B'(y)\epsilon$$

Observe that

$$D^{MAC} = -\frac{\Delta B}{B(y)} \cdot \frac{1 + y}{\Delta y}$$

$$D^{MAC} = -B'(y) \cdot \frac{1 + y}{B(y)}$$

$$B'(y) = -\frac{B(y)D^{MAC}}{1 + y}$$

Then

$$\Delta B \approx -\frac{B(y)D^{MAC}}{1 + y}\epsilon$$

$$D^{MAC} \approx \frac{\Delta B}{\epsilon} \cdot \frac{1 + y}{B(y)}$$

Suppose now that we wish to create a bond portfolio that hedges against shifts in the yield. This hedging strategy is called **immunization** or **duration matching**. Consider the two-asset bond portfolio

$$P = B_1 + NB_2$$

We want to find which position we want to take in the second bond to hedge against interest rate risk. The hedge requires that the price change following a change in the yield of ϵ

$$\Delta P = 0$$

Therefore,

$$\Delta B_1 + N\Delta B_2 = 0$$

$$N = -\frac{\Delta B_1}{\Delta B_2}$$

Using the approximation for the price change gives

$$N \approx -\frac{-\frac{B_1(y_1)D_1^{MAC}}{1+y_1}\epsilon}{-\frac{B_2(y_2)D_2^{MAC}}{1+y_2}\epsilon}$$

$$N \approx -\frac{B_1(y_1)}{B_2(y_2)} \cdot \frac{D_1^{MAC}}{D_2^{MAC}} \cdot \frac{1+y_2}{1+y_1}$$

This is the position in the second bond that hedges the bond portfolio against yield curve shifts.

Convexity

The issue with the duration matching strategy above is that it is based on a linearization (a first order linear approximation) of the duration. Since changes in the bond price changes for when the yield changes, so will the duration. Essentially, we are basing the hedge on durations that are only approximate. This approximation works well when ϵ is small, but as ϵ gets larger the approximation becomes less accurate.

We account for this second order effect of the bond price by using the **convexity**. The convexity measures the change in bond price for different yields. Formally, this is the second derivative of the bond price with respect to the yield, divided by the bond price.

$$C = \frac{1}{B(y)} \frac{\partial^2 B(y)}{\partial y^2}$$

$$C = \frac{1}{B(y)} \left[\sum_{i=1}^{n} \frac{i(i+1)}{m^2} \cdot \frac{\frac{C}{m}}{\left(1+\frac{y}{m}\right)^{i+2}} + \frac{n(n+1)}{m^2} \cdot \frac{M}{\left(1+\frac{y}{m}\right)^{n+2}} \right]$$

We divide by the bond price to get a percentage change rather than the level change.

The second order Taylor approximation is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$$

Let $f(x_0) = B(y)$ and $x = y + \epsilon$ where ϵ is a small number. Then

$$B(y + \epsilon) \approx B(y) + B'(y)\epsilon + \frac{1}{2}B''(y)\epsilon^2$$

Using that

$$C = \frac{1}{B(y)}\frac{\partial^2 B(y)}{\partial y^2}$$

$$\frac{\partial^2 B(y)}{\partial y^2} = B(y)C$$

We arrive at

$$B(y + \epsilon) \approx B(y) + B'(y)\epsilon + \frac{1}{2}B(y)C \cdot \epsilon^2$$

$$\Delta B \approx B'(y)\epsilon + \frac{1}{2}B(y)C \cdot \epsilon^2$$

Chapter 2: Forwards and Futures

Forwards

A ***forward contract*** is an agreement between two parties to trade a specified quantity of a specified good at a specified price on a specified date in the future. Forward are primarily used for hedging purposes, but they can also be used as speculative instrument. An important feature of forward contracts is that they are costless to enter. That means that none of parties behind the contract will pay anything at time 0.

This is the basic terminology used for forwards

- Buyer of the forward contract has a ***long*** position in the contract
- Seller of the forward contract has a ***short*** position in the contract
- The date in which the trade of the good will take place is the ***maturity date***
- The good is called the ***underlying asset***
- The price in the contract is called the ***delivery price***

Let the delivery price be denoted as $F_{0,T}$ and the maturity date T. Then, let S_T denote the value of the underlying at time T. The payoff for the long position of a forward contract is

$$S_T - F_{0,T}$$

For the short position, the payoff becomes

$$F_{0,T} - S_T$$

Therefore, this is a zero-sum trade (one party's gain is the other party's losses). It is also clear that the payoff structure is linear, so forward contracts are linear derivatives.

Forward Pricing on Financial Assets

Forwards are priced using **replication**. Replication is a concept that is based on the idea that the price of a derivative (such as a forward) must be the same as creating the same outcome/payoff using other securities. To be able to price derivatives using replication, we will need a no arbitrage assumption. The no arbitrage assumption states that arbitrage opportunities cannot persist in a market.

Now, consider a long position in the forward. The forward will involve no cash flows until time T in which the delivery price $F_{0,T}$ must be paid. From time 0, the cost of this strategy must then be $PV(F_{0,T})$. This is called the **prepaid forward price** because it is the price you would pay today to receive the stock at time T. An alternative notation for the prepaid forward is $F_{0,T}^{P} = PV\left(F_{0,T}\right)$

To replicate this strategy, we first buy the underlying asset at price S_0. This purchase will give us S_T at time T. However, the asset purchase may also come with holding benefits and/or costs such as dividends, coupons, or storage costs. Therefore, purchasing an asset may come with implicit costs excluding the asset price. Let M be the present value of net holding costs. Then,

$$M = PV(Holding\ Costs) - PV(Holding\ Benefits)$$

Therefore, the total cost of this strategy must be

$$S_0 + M$$

In a no-arbitrage market we will then require

$$PV\left(F_{0,T}\right) = S_0 + M$$

With continuous compounding, we solve for the delivery price and get

$$F_{0,T} = (S_0 + M)e^{-rT}$$

This is the fundamental theoretical pricing equation for forward contracts.

- If $PV\left(F_{0,T}\right) > S_0 + M$
  - Sell forward, purchase asset at spot
  - This is a cash-and-carry arbitrage
- If $PV\left(F_{0,T}\right) < S_0 + M$
  - Buy forward, short asset at spot
  - This is a reverse cash-and-carry arbitrage

We can now derive a general pricing formula for forward contracts. Consider an asset that pays a continuous dividend yield δ. Suppose we buy one unit of the stock at S_0. If the dividends are reinvested into the asset continuously, one share today will produce $e^{\delta T}$ shares at time T. Therefore, if we want one share at time T, we will have to purchase $e^{-\delta T}$ shares at time 0.

If we go back to the derivation of the theoretical forward price, we see that the no-arbitrage condition will now require

$$F_{0,T}^{P} = S_0 e^{-\delta T}$$

$$F_{0,T}^{P} e^{rT} = S_0 e^{(r-\delta)T}$$

$$F_{0,T} = S_0 e^{(r-\delta)T}$$

Which is the forward price for an asset that pays a continuous dividend. This relationship can be used to derive the **forward premium**, which is the ratio of the forward price to the spot price.

$$\frac{F_{0,T}}{S_0} = e^{(r-\delta)T}$$

Since $\ln\left(\frac{F_{0,T}}{S_0}\right)$ is the percentage representation for the forward premium over $[0, T]$, dividing it by T will give the **annualized forward premium**. We can then see that

$$\frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right) = r - \delta$$

Forward Pricing on Currencies

When pricing forward contracts on currencies, we must consider that currencies earn interest. Therefore, one unit of a foreign currency at time 0 will be more than one unit of the same foreign currency at time T. Therefore, the replicating strategy must factor that currencies earn interest yield.

Consider an investor who enters a currency forward. The investor will pay $F_{0,T}$ in the domestic currency (NOK) at time T for one unit of the foreign currency (USD). To replicate this outcome, we cannot buy one unit of the foreign currency today and hold to T because by then it would have grown to more than one unit.

Let x_0 be the current price in NOK for one unit of USD, that is, simply the exchange rate NOK/USD. Then, by going long on a currency forward we will pay F NOK for 1 USD at time T. The cost of this strategy in NOK at $t = 0$ must be $F_{0,T} \cdot PV(1\ NOK)$.

Now, we need to replicate the forward payoff. To receive 1 USD at time T, we must purchase the present value of 1 USD keeping in mind that we are discounting with the US currency yield. The present value of 1 USD at time 0 is $x_0 \cdot PV(1\ USD)$. These strategies must yield the same costs, therefore

$$F_{0,T} \cdot PV(1\ NOK) = x_0 \cdot PV(1\ USD)$$

$$F_{0,T} = x_0 \cdot \frac{PV(1\ USD)}{PV(1\ NOK)}$$

$$F_{0,T} = x_0 \cdot \frac{e^{-r_{US}T}}{e^{-r_{NOK}T}}$$

$$F_{0,T} = x_0 e^{(r_{NOK} - r_{US})T}$$

To clean up the notation, let x_0 be the spot exchange rate, r be the domestic interest rate and r_f the foreign rate. Then, to receive one unit of the foreign currency at time T, the forward price in the domestic currency will be

$$F_{0,T} = x_0 e^{(r-r_f)T}$$

This is the currency forward pricing formula. The interpretation is that the forward price $F_{0,T}$ is the exchange rate you lock in today which will hold at time T. Therefore, currency forwards can be used as a hedging instrument for hedging exchange rate risk.

Forward Pricing on Commodities

Forward contracts on commodities share similar characteristics with forward contracts on financial assets. However, when pricing commodity forwards, some considerations must be made

- **Storage costs**. The party that is long on the forward contract must pay the counterparty for the storage costs for holding the commodity.
- **Carry markets**. A market for which the forward price compensates commodity owners for storage costs is a carry market.
- **Lease rate**. A commodity short seller may have to compensate the owner commodity for lending the commodity.
- **Convenience yield**. The owner of a commodity may receive benefits from holding the commodity. These benefits include higher profits on commodities when there are shortages or when there is an unexpected need for additional production inputs. Such benefits may be reflected in forward prices as a convenience yield.

Suppose that we have a commodity with discount rate α. Then, the price today to receive one unit of the commodity in the future is

$$F_{0,T}^P = E_0[S_T]e^{-\alpha T}$$

Since $F_{0,T} = F_{0,T}^P e^{rT}$, we can write

$$F_{0,T} = E_0[S_T]e^{-(\alpha-r)T}$$

Or alternatively

$$F_{0,T}e^{-rT} = E_0[S_T]e^{-\alpha T}$$

This relationship states the forward price discounted at the risk-free rate must equate the present value for the unit of the commodity.

The Lease Rate for Commodity Short Sales

Suppose now that we have an investor who wishes to short sell the commodity and is willing to pay a continuous lease rate δ_L to the commodity owner. From the perspective of the short seller, he will receive the current commodity price S_0 today and return the commodity with a lease payment at time T. The net present value of this trade is

$$NPV = -E_0[S_T]e^{-\alpha T}e^{\delta_L T} + S_0$$

This net present value must be zero, otherwise an arbitrage profit can be made from short selling the commodity.

Then,

$$S_0 = E_0[S_T]e^{-\alpha T}e^{\delta_L T}$$

The leasing rate the commodity owner will require that is consistent with an arbitrage-free market is

$$\delta_L = \alpha - \frac{1}{T}\ln\left(\frac{E_0[S_T]}{S_0}\right)$$

From the forward price equation, we have

$$E_0[S_T] = \left(F_{0,T}\right)^{(\alpha-r)T}$$

Inserting it in the lease rate equation gives

$$\delta_L = \alpha - \frac{1}{T}\ln\left(\frac{\left(F_{0,T}\right)^{(\alpha-r)T}}{S_0}\right)$$

$$\delta_L = \alpha - \alpha + r - \frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right)$$

$$\delta_L = r - \frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right)$$

Pricing Commodity Forwards with Storage Costs

Suppose that the future value of the storage costs for one unit of the commodity is $\lambda(0,T)$ and that the storage cost is paid at time T. If a commodity owner is presented with the opportunity to either sell the commodity today or sell it forward with expiration at time T, he will only sell the commodity at time T if the present value of the time T forward price and the storage costs exceeds the current price S_0. The cash-and-carry strategy is

Transaction	$t = 0$	$t = T$
Long commodity	$-S_0$	S_T
Pay storage cost	0	$-\lambda(0,T)$
Short commodity forward	0	$F_{0,T} - S_T$
Borrow S_0	S_0	$-S_0 e^{rT}$
Total	0	$F_{0,T} - \left(S_0 e^{rT} + \lambda(0,T)\right)$

In equilibrium, we require

$$F_{0,T} = S_0 e^{rT} + \lambda(0,T)$$

If the storage cost is paid continuously, the forward price becomes

$$F_{0,T} = S_0 e^{(r+\lambda)T}$$

Convenience Yields

Holding commodities may also come with benefits. These benefits include higher prices on commodities when there are shortages or when there is an unexpected need for additional production inputs. Commodities may also come with other nonmonetary benefits. Such benefits from holding commodities are reflected by a ***convenience yield***. It is not an observable yield in the market, but it is an important factor for pricing commodity forwards.

Convenience yields make short-selling commodities more difficult. Suppose we have a commodity with no storage costs. Let $\hat{F}$ be the observed forward price in the market. If $\hat{F} > F$, the forward is overpriced relative to the theorical price. We would then expect a price correction as investors short the commodity and purchase the forward contract. Therefore,

$$\hat{F} \leq S_0 e^{rT}$$

Let c be the annualized convenience yield. Then the present value for one unit of the commodity at time 0 is

$$e^{-cT}$$

To enter the arbitrage position, the investor does the following

Transaction	$t = 0$	$t = T$
Long forward	0	$S_T - \hat{F}$
Short PV of commodity	$S_0 e^{-cT}$	$-S_T$
Invest at risk-free rate	$-S_0 e^{-cT}$	$S_0 e^{(r-c)T}$
Total	0	$S_0 e^{(r-c)T} - \hat{F}$

The investor makes an arbitrage profit provided

$$S_0 e^{(r-c)T} - \hat{F} > 0$$

Therefore, under the assumption of no arbitrage, we would require

$$S_0 e^{(r-c)T} \leq \hat{F}$$

In total, we must require

$$S_0 e^{(r-c)T} \leq \hat{F} \leq S_0 e^{rT}$$

This makes it clear that commodity forwards results in a price interval in which the investor cannot make an arbitrage profit from rather than one single price. If we now assume that the commodity requires a continuous storage cost λ, we can write

$$S_0 e^{(r+\lambda-c)T} \leq \hat{F} \leq S_0 e^{(r+\lambda)T}$$

Backwardation and Contango

We say that the forward market is in ***contango*** when the forward prices exceed the spot prices

$$F > S_0$$

If there are no convenience yields, contango is the normal situation predicted by the pricing equation.

When forward prices are lower than the spot price, we say that the market is in ***backwardation***. This is when

$$F < S_0$$

This typically occurs in markets with high convenience yields.

Forward Rate Agreements

Forward rate agreements (FRAs) are forward contracts written on interest rates. FRAs allow investors to lock in an interest rate k for borrowing or lending a specified principal amount P over a period $[t_1, t_2]$. This is called a $t_1 - t_2$ FRA because it begins in t_1 months and ends in t_2 months.

The long position holder in the FRA will receive the difference between a reference rate R and the agreed-upon fixed interest rate k, that is, $R - k$. If this difference is negative, then the interpretation is that the long position must make a payment to the short position. Let d be the investment period in days. Assume that the payment is settled at the end of the contract, t_2. Then the payoff to the long position will be

$$P \cdot \frac{(R-k) \cdot \frac{d}{360}}{1 + R \cdot \frac{d}{360}}$$

Valuing FRAs

When the FRA is initiated, the fixed rate k is chosen such that the contract has zero value for both parties. This interest rate is what is referred to as the **price of the FRA**. We can derive this interest rate by replication.

We know that the payoff to the long position is

$$P \cdot \frac{(R-k)\cdot \frac{d}{360}}{1+R\cdot \frac{d}{360}}$$

$$\frac{\left(PR\frac{d}{360} - Pk\frac{d}{360}\right)}{1+R\cdot \frac{d}{360}}$$

Add and subtract the principal in the numerator

$$\frac{\left(P + PR\frac{d}{360} - Pk\frac{d}{360} - P\right)}{1+R\cdot \frac{d}{360}}$$

Separate the terms

$$\frac{P + PR\frac{d}{360}}{1+R\cdot \frac{d}{360}} - \frac{Pk\frac{d}{360} - P}{1+R\cdot \frac{d}{360}}$$

This can be written as

$$P - P\cdot \frac{1+k\cdot \frac{d}{360}}{1+R\cdot \frac{d}{360}}$$

The first term P is a certain cash inflow at time t_1. The second term is an outflow that is uncertain because the reference rate R is not determined until t_2. Regardless of the outcome of R, if the second term is invested at R at time t_1, the amount will grow to

$$P\cdot \frac{1+k\cdot \frac{d}{360}}{1+R\cdot \frac{d}{360}} \cdot \left(1+R\cdot \frac{d}{360}\right) = P\left(1+k\cdot \frac{d}{360}\right)$$

The FRA contract will then consist of two cash flows. A certain inflow of P at t_1 and a certain outflow of $P\left(1+k\cdot \frac{d}{360}\right)$ at time t_2. Let $B(T)$ denote the present value of 1 receivable at time T.

Then, the present value of these two cash flows is

$$PV = B(t_1)P - B(t_2)P\left(1 + k \cdot \frac{d}{360}\right)$$

Since an FRA has a zero value at initiation, we set the present value to zero and solve for k. This gives

$$k = \frac{B(t_1) - B(t_2)}{B(t_2)} \cdot \frac{360}{d}$$

Futures

Futures contracts are essentially forward contracts that are traded on an exchange. Although forwards and futures appear to be alike, there are a few differences.

- Futures are settled daily, rather than at expiration. This daily settlement is called **marking-to-market** and can lead to differences in pricing of futures and an otherwise identical forward.
- Futures are liquid. It is possible to offset the obligation from the future by entering an opposite position.
- Futures contracts are standardized whereas OTC forward contracts can be customized.
- Futures contracts tend to have a lower credit risk due to the daily settlement.

Since futures positions are marked-to-market, forward and future prices will often differ. This is because interest is earned on the proceeds from the mark-to-market.

Chapter 3: Swaps

Commodity Swaps

A **swap** is a periodic exchange of cash flows under some specified rules. One subset of swap contracts are **commodity swaps**. Commodity swaps are swap contracts in which a commodity is swapped for cash. The swap contract must specify which commodity is being swapped, how many units are being swapped and how many settlements are going to be made. This unit exchange of commodities is called the **notional amount** of the swap and is used to determine the magnitude of payments. Suppose for instance that the notional amount is 100 000 barrels of oil. Then all swap payments are based on an exchange of 100 000 barrels.

We can now derive the swap price. Suppose that there are n settlements occurring on dates $t_i, i = 1, \dots, n$. The price of a zero-coupon bond maturing on date t_i is

$$P(0, t_i) = \frac{1}{\left(1 + r(0, t_i)\right)^{t_i}}$$

If the buyer of the swap were enter a series of forward contracts to purchase one unit on each date t_i, the present value of this position is simply the present value of each forward price, discounted with the appropriate discount rate. These forward contracts replicate the commodity swap, so the present value of these forward contracts must yield **prepaid swap price** which is the price you would pay today for the swap.

$$S_{0,T}^{P} = \sum_{i=1}^{n} F_{0,t_i} P(0, t_i)$$

The **fixed swap price**, R, is the price we would pay each year to obtain the commodity. By a no-arbitrage argument, this agreement must have the same present value as the prepaid swap price.

$$\sum_{i=1}^{n} R P(0, t_i) = \sum_{i=1}^{n} F_{0,t_i} P(0, t_i)$$

Which solves for

$$R = \frac{\sum_{i=1}^{n} F_{0,t_i} P(0, t_i)}{\sum_{i=1}^{n} P(0, t_i)}$$

Now, suppose instead that the buyer wishes to enter a swap in which the quantity of the commodity varies over time. To do so, we simply adjust for the number of quantities to be purchased at time t_i. Let Q_{t_i} denote the quantity of the commodity to be delivered at time t_i. Then, the prepaid swap price must be

$$S_{0,T}^{P} = \sum_{i=1}^{n} Q_{t_i} F_{0,t_i} P(0, t_i)$$

The fixed swap price equivalent entails purchasing Q_{t_i} units for R in each period. Then,

$$\sum_{i=1}^{n} RQ_{t_i}P(0,t_i) = \sum_{i=1}^{n} Q_{t_i}F_{0,t_i}P(0,t_i)$$

The fixed swap price is then

$$R = \frac{\sum_{i=1}^{n} Q_{t_i}F_{0,t_i}P(0,t_i)}{\sum_{i=1}^{n} Q_{t_i}P(0,t_i)}$$

Interest Rate Swaps

An **interest rate swap** is a swap contract where the cash flows are based on interest rates. To motivate their use, consider a firm with a floating rate debt that would prefer to have a fixed-rate debt. The firm could convert the floating rate debt to fixed rate debt in three ways:

- Retire floating rate debt and issue a fixed-rate debt in its place
  - Upside: Firm now has fixed-rate debt
  - Downside: Costly
- Enter a series of FRAs to lock in a fixed rate for each payment
  - Upside: Rates are known in advance, and are so “fixed”
  - Downside: The “fixed” rate will vary each year
- Enter an interest rate swap
  - Receive a floating rate and pay a fixed rate, then use the floating rate to pay back the debt
  - The loan becomes fixed

We can now construct an interest swap in a similar fashion as the commodity swap. Suppose the buyer of the swap wishes to pay **fixed-for-floating**, that is, paying the fixed rate R and receive the floating rate. Let the time 0 implied forward rate between t_{i-1} and t_i be the floating rate. Generally, this is

$$r_t(t_{i-1}, t_i) = \frac{P(0,t_{i-1})}{P(0,t_i)} - 1$$

$$r_t(t_{i-1}, t_i) = \frac{P(0,t_{i-1}) - P(0,\ t_i)}{P(0,t_i)}$$

The present value of the cash inflows the buyer receives is

$$\sum_{i=1}^{n} P(0,t_i) \cdot r_0(t_{i-1},t_i)$$

The present value of the cash inflows the seller receives is

$$\sum_{i=1}^{n} P(0,t_i)R$$

The interest rate swap should have a value of zero initially, by construction. Therefore, the present value of inflows and outflows must be zero. So,

$$\sum_{i=1}^{n} P(0,t_i) \cdot r_0(t_{i-1},t_i) - \sum_{i=1}^{n} P(0,t_i)R = 0$$

$$\sum_{i=1}^{n} P(0,t_i) \cdot r_0(t_{i-1},t_i) = \sum_{i=1}^{n} P(0,t_i)R$$

$$R = \frac{\sum_{i=1}^{n} P(0,t_i) r_0(t_{i-1},t_i)}{\sum_{i=1}^{n} P(0,t_i)}$$

$$R = \frac{\sum_{i=1}^{n} P(0,t_i)\left(\frac{P(0,t_{i-1})}{P(0,t_i)} - 1\right)}{\sum_{i=1}^{n} P(0,t_i)}$$

$$R = \frac{\sum_{i=1}^{n}\big(P(0,t_{i-1}) - P(0,t_i)\big)}{\sum_{i=1}^{n} P(0,t_i)}$$

Note that

$$\sum_{i=1}^{n}\big(P(0,t_{i-1}) - P(0,t_i)\big) = P(0,t_0) - P(0,t_n)$$

$$R = \frac{P(0,t_0) - P(0,t_n)}{\sum_{i=1}^{n} P(0,t_i)}$$

If the interest rate swap starts immediately, that is, $t_0 = 0$, we must have

$$P(0,t_0) = \frac{1}{\left(1 + r(0,t_0)\right)^{t_0}} = 1$$

Otherwise we would receive an instant positive cash flow with no cost. Therefore,

$$R = \frac{1 - P(0,t_n)}{\sum_{i=1}^{n} P(0,t_i)}$$

Chapter 4: Options

An **option** is a financial derivative that gives its holder the right to buy or sell a specified quantity of a specified asset at a specified price on some agreed-upon date(s). Option terminology is listed below

Term	**Interpretation**
Call option	Right to buy the underlying asset
Put option	Right to sell the underlying asset
Expiration/Maturity date	Date on which the right expires
Strike/Exercise price	Price at which the right may be exercised
Long position/Holder/Buyer	Party that holds the right
Short position/Writer/Seller	Party with a contingent obligation
American-style option	Right may be exercised any time before maturity
European-style option	Right may be exercised only at maturity
Bermudan-style option	Right may be exercised before maturity, but only at pre-specified dates

We need some notation as well

- S_t: Current price of the underlying asset at time t
- T: Maturity date
- K: Strike price
- C: Current call price
- P: Current put price

The gross payoffs (before we factor the purchase/sale of the option) for the standard options are

Long put	$\max(0, S_T - K)$
Short put	$-\max(0, S_T - K)$
Long call	$\max(0, K - S_T)$
Short call	$-\max(0, K - S_T)$

Price Bounds on Calls

Option prices will depend on several factors. The prices are different for calls and puts, but also across option styles, exercise price and time to maturity.

We start by deriving bounds on call prices. Consider a call option. A reasonable upper bound on its price is the value of the underlying asset. There is no reason that a call option on the asset should be worth more than the underlying asset itself. Therefore,

$$C \leq S_0$$

It is also clear that the option price cannot be negative. In that case, the investor would be paid for the right to throw away the option for free. Therefore,

$$C \geq 0$$

When the option is of American style, another lower bound is

$$C_A \geq S_0 - K$$

Since American calls can be exercised at any time before T. If exercised immediately, the investor receives $S_0 - K$. If this relationship does not hold, the investor can make an arbitrage profit by purchasing the call and exercising it immediately.

The final lower bound will hold for both European and American calls. Consider a European call on a non-dividend paying asset. We construct the following two portfolios

Transaction	$t = 0$	$t = T, S_T < K$	$t = T, S_T > K$
Long call	$-C_E$	0	$S_T - K$
Long underlying	$-S_0$	S_T	S_T
Borrow $PV(K)$	$PV(K)$	$-K$	$-K$

We see that the long call will strictly dominate the underlying + borrowing portfolio. Therefore,

$$C_E \geq S - PV(K)$$

Now, consider the case when the underlying asset also pays dividends. Assume for simplicity that this dividend is paid at T as well. We then construct the same portfolio as above, but now with another borrowing element

Transaction	$t = 0$	$t = T, S_T < K$	$t = T, S_T > K$
Long call	$-C_E$	0	$S_T - K$
Long underlying	$-S_0$	$S_T + D$	$S_T + D$
Borrow $PV(K) + PV(D)$	$PV(K) + PV(D)$	$-K - D$	$-K - D$

For the same reason as above, the call will strictly dominate. Now, the lower bound becomes

$$C_E \geq S - PV(K) - PV(D)$$

We can use these inequalities to determine one single expression for each option style

- $C_E \geq \max\big(0, S_0 - PV(K) - PV(D)\big)$
- $C_A \geq \max\big(0, S - K, S - PV(K) - PV(D)\big)$

Price Bounds on Puts

Now, we turn to put options. If the price of the underlying cannot become negative, the maximum payoff on a put option is K. So, the upper bound becomes

$$P \leq K$$

When the put is of European style, the investor will have to wait until time T to exercise the put. Therefore, the maximum profit of K at time T is only worth $PV(K)$ today. So,

$$P_E \leq PV(K)$$

For the same reason as with the call options, put options cannot have a negative price.

$$P \geq 0$$

To prevent arbitrage when the put is American, the put must cost at least as much as the payoff from immediate exercise

$$P_A \geq K - S_0$$

Now, we derive the last lower bound. Consider the case when the underlying asset also pays dividends at time T. We then create the following two portfolios

Transaction	$t = 0$	$t = T, S_T < K$	$t = T, S_T > K$
Long put	$-P_E$	$K - S_T$	0
Short underlying	S_0	$-S_T - D$	$-S_T - D$
Lend $PV(D) + PV(K)$	$-PV(D) - PV(K)$	$D + K$	$D + K$

The put option strictly dominates the other portfolio. Therefore,

$$P_E \geq PV(D) + PV(K) - S_0$$

Lastly, American puts must always cost more than European puts. So,

$$P_A \geq P_E$$

To summarize

$$K \geq P_E \geq \max(0, PV(K) + PV(D) - S_0)$$

$$K \geq P_A \geq \max(0, K - S_0, PV(K) + PV(D) - S_0)$$

Insurance Values on Options

Holding options provide investors with protections against undesirable price movements. The value of this protection is the insurance value of the option. Consider the portfolio used for deriving price bounds on call options.

Transaction	$t = 0$	$t = T, S_T < K$	$t = T, S_T > K$
Long call	$-C_E$	0	$S_T - K$
Long underlying	$-S_0$	$S_T + D$	$S_T + D$
Borrow $PV(K) + PV(D)$	$PV(K) + PV(D)$	$-K - D$	$-K - D$

As we saw earlier, when $S_T < K$ the synthetic forward results in a payoff of $S_T - K < 0$ while the call provides a payoff of 0. This insurance that the call option provides will be reflected in its price. Therefore, a measure of the insurance value for the call option is the difference between the two portfolios

$$IV(C) = C - [S_0 - PV(K) - PV(D)]$$

We can find the insurance value for a put as well. Using the same principle, consider the synthetic short forward portfolio

Transaction	$t = 0$	$t = T, S_T < K$	$t = T, S_T > K$
Long put	$-P_E$	$K - S_T$	0
Short underlying	S_0	$-S_T - D$	$-S_T - D$
Lend $PV(D) + PV(K)$	$-PV(D) - PV(K)$	$D + K$	$D + K$

The difference in the price between the two portfolios must be the insurance value. Therefore,

$$IV(P) = P - [PV(K) + PV(D) - S_0]$$

Call Options and Strikes

We now examine option prices for different values of K.

Consider two call options with strike prices K_1 and K_2. If $K_2 > K_1$, then

$$C(K_1) \geq C(K_2)$$

This is logical because the payoff at K_1 is larger for any payoff in which $S_T > K_1$. A formal proof can be done by assuming otherwise and setting up a bull spread.

We now derive a maximum difference on the two call options. The maximum additional payoff from using the K_1 strike call instead of the K_2 strike call is $K_2 - K_1$. Then, if the option is of American style, we can write

$$C_A(K_1) - C_A(K_2) \leq K_2 - K_1$$

For European style call options, the payoff $K_2 - K_1$ cannot be exercised until time T. Therefore,

$$C_E(K_1) - C_E(K_2) \leq PV(K_2 - K_1)$$

The final price restriction relates any three call options that differ only in strike prices. Define

$$w = \frac{K_3 - K_2}{K_3 - K_1}$$

By going long on w units of the K_1 strike call, $1 - w$ units of the K_3 strike call and short on one unit of the K_2 strike call, we guarantee a non-negative payoff. Therefore, the price on this strategy must also be positive. So,

$$wC(K_1) + (1 - w)C(K_3) \geq C(K_2)$$

Put Options and Strikes

Let $K_1 < K_2$. Then we must require that

$$P(K_1) < P(K_2)$$

This restriction makes sense as the K_2 strike put will provide a larger payoff than the K_1 strike put for any given price of the underlying asset, provided it is non-zero. A formal proof can be done by assuming otherwise and entering a bear spread.

The second restriction is related to the maximum price difference between two puts that differ in strike prices. We use a similar argument as we did for the call options. The maximum additional payoff for the K_2 strike put is $K_2 - K_1$. Therefore,

$$P_A(K_2) - P_A(K_1) \leq K_2 - K_1$$

$$P_E(K_2) - P_E(K_1) \leq PV(K_2 - K_1)$$

Lastly, define

$$w = \frac{K_3 - K_2}{K_3 - K_1}$$

And assume that $K_1 < K_2 < K_3$. By going long on w units of the K_1 strike put, $1 - w$ units of the K_3 strike put and short on one unit of the K_2 strike put, we guarantee a non-negative payoff. Therefore, the price on this strategy must also be positive. So,

$$wP(K_1) + (1 - w)P(K_3) \geq P(K_2)$$

Call Prices and Time

Consider two call options that differ in time to maturity but are otherwise identical. We want to examine the call prices when the expiration dates are different. Suppose we have two American call options. If $T_1 < T_2$, then

$$C_A(T_2) > C_A(T_1)$$

This restriction is intuitive as longer expiration dates increase the chance of the option yielding a positive payoff.

We can also show that this result holds for European call options, provided the asset does not pay any dividends between T_1 and T_2.

Recall that $C_E \geq S_0 - PV(K)$ and consider two European call options with expiration dates T_1 and T_2 where the underlying does not pay any dividend. This is equivalent with $PV(D) = 0$. If we are at date T_1, then the payoff on the call with maturity T_1 is worth $\max(0, S_{T_1} - K)$ and the call with maturity T_2 is worth *at least* $\max\left(0, S_{T_1} - PV(K)\right)$. Since $PV(K) \leq K$, the payoff on the T_2 strike call is always equal or greater than the T_1 strike call. Therefore, its price must be higher. Thus, if $T_2 > T_1$, then

$$C_E(T_2) \geq C_E(T_1)$$

Now, if there is a dividend payment after T_1, this will lower the value of the T_2 strike call without affecting then T_1 strike call price. So, the relationship above may not hold.

Put Prices and Time

If we use the same argument as in the case with the American calls, it is logical that we would require

$$P_A(T_1) \leq P_A(T_2)$$

Provided that $T_1 < T_2$. However, this relationship may not necessarily hold for European puts.

Decomposing Option Prices

Option prices can be decomposed into four parts that will help our understanding early exercise, which will be discussed later. Consider the insurance value for the call option

$$IV(C) = C - [S_0 - PV(K) - PV(D)]$$

Rewrite for the call price

$$C = S_0 - PV(K) - PV(D) + IV(C)$$

Add and subtract the strike price

$$C = \underbrace{(S_0 - K)}_{Intrinsic\ Value} + \underbrace{\big(K - PV(K)\big)}_{Time\ Value} + \underbrace{\big(IV(C)\big)}_{Insurance\ Value} + \underbrace{\big(-PV(D)\big)}_{Payout}$$

- The intrinsic value measures the call price to the actual payoff from the option
- The time value of the call measures the interest savings we obtain from this deferred purchase
- The insurance value measures the value of downside protection
- The payout measures the negative impact of dividend payments on the call option

We can derive a similar decomposition of put prices. The relationship is

$$P = (K - S) - \big(K - PV(K)\big) + IV(P) + PV(D)$$

Which is slightly different from the case of call options.

Early Exercise Optimality

The holder of an American call option has three possibilities open at any point

- Exercise the call immediately and receive $S_0 - K$
- Sell the call and receive C_A
- Do nothing

The optimality of early exercise is derived by comparing the first case to the last two. Intuitively, the investor should never exercise the option if $C_A > S_0 - K$ because he would receive a larger profit from selling the option instead. When the underlying asset pays no dividends, the value of the call option is given by

$$C_A = (S_0 - K) + \big(K - PV(K)\big) + IV(C)$$

Then, the difference between selling and exercising the call is

$$C_A - (S_0 - K) = \big(K - PV(K)\big) + IV(C) > 0$$

Therefore, if the underlying asset does not pay any dividends, early exercise is never optimal. The intuition is that exercising early results in giving up two things. First, you lose out on time value because you could have bought the asset at K at a later point in time rather than today. Since K today is worth more than K at a later point in time, you lose out on interest earnings. Second, by exercising the option today you will also lose out on the protection reflected in the insurance value. Since the asset does not pay out any dividends, you receive no benefits from exercising the option early.

Let us now consider the case where the asset pays out dividends. Then the difference between the two strategies become

$$C_A - (S_0 - K) = \big(K - PV(K)\big) + IV(C) - PV(D)$$

Since the payout term is negative, we cannot for sure guarantee that early exercise is suboptimal. Therefore, early exercise may be optimal. To build on the intuition above, when dividends are included in the asset, the option holder receives some benefit from exercising early which might make early exercise optimal. The gains equate the losses when

$$\big(K - PV(K)\big) + IV(C) = PV(D)$$

As is seen from the decomposition of the call price, early exercise is more likely to be optimal if

- Dividends are large
    - Larger benefits from early exercise
- Volatility is low
    - Low insurance value
- Low interest rates
    - Low time value

Now, we turn to American puts. Early exercise for American puts is suboptimal if $P_A > K - S_0$. First, we assume that the underlying asset pays no dividends. Then, the price difference between exercising early and selling the option is

$$P_A - (K - S) = -\big(K - PV(K)\big) + IV(P)$$

Since the first term is negative and second term is positive, we cannot rule out that exercising early is suboptimal. The intuition that drives this result is that delaying exercise of the put means receiving the strike at a later point in time, which results in a loss of interest earnings that could have been earned on the strike. However, delaying exercise retains the insurance value of the put which retains the possibility of selling the stock at a higher price later. Early exercise is more likely to be optimal if

- Volatility is low
    - Low insurance value
- Interest rates are high
    - High time value

If the underlying pays dividends, the difference instead becomes

$$P_A - (K - S) = -\big(K - PV(K)\big) + IV(P) + PV(D)$$

The additional factor to consider when the underlying pays dividends is that delaying exercise also comes with the benefits of obtaining dividends. Therefore, large dividend payments will motivate the investor to delay exercise.

The Put-Call Parity

We can now examine the relationship between calls and puts by deriving the **Put-Call parity**. Consider a European put and call option that are otherwise equal.

Consider two portfolios

- Portfolio A
  - Long call + Investment of $PV(K)$
- Portfolio B
  - Long put + Long asset

Transaction	$t = 0$	$t = T, S_T < K$	$t = T, S_T > K$
Portfolio A	$-C_E - PV(K)$	$K + 0$	$S_T - K + K = S_T$
Portfolio B	$-P_E - S_0$	$K - S_T + S_T = K$	S_T

No arbitrage requires the cost of these two portfolios to be the same. Therefore,

$$C_E + PV(K) = P_E + S_0$$

If this condition is broken there will exist arbitrage opportunities

- If $C_E + PV(K) > P_E + S_0$
  - Short portfolio A
  - Long portfolio B
- If $C_E + PV(K) < P_E + S_0$
  - Short portfolio B
  - Long portfolio A

We can extend the parity condition for options on dividend-paying assets. Construct the following portfolio

- Portfolio A
  - Long call + Investment of $PV(K)$ + Investment of $PV(D)$
- Portfolio B
  - Long put + Long asset

Transaction	$t = 0$	$t = T, S_T < K$	$t = T, S_T > K$
Portfolio A	$-C_E - PV(K)$ $- PV(D)$	$K + D$	$S_T + D$
Portfolio B	$-P_E - S_0$	$K + D$	$S_T + D$

This changes the cost of portfolio A to $C_E + PV(K) + PV(D)$. Then, the parity instead becomes

$$C_E + PV(K) + PV(D) = P_E + S_0$$

If the options are of American style, we cannot compare the portfolio values at maturity because American options can be exercised prior to that. In this case, we are not able to derive an exact parity on American options. However, we can derive two inequality-based relationships.

Suppose that we have two American options where the underlying asset does not pay dividends. Then, consider two portfolios

- Portfolio A
    - Long call + Investment of $PV(K)$
- Portfolio B
    - Long put + Long asset

Portfolio A has an initial cost of $C_A + PV(K)$ while portfolio B's initial cost is $P_A + S$. Now, we must note two things

- American calls on non-dividend paying assets will never be exercised early. Then, $C_A = C_E$
- Exercising an American put early may be optimal. Then, $P_A \geq P_E$

We can then write

$$C_A + PV(K) \leq P_A + S_0$$

Now, consider the two portfolios

- Portfolio A*
    - Long call + Investment of K rolled over at the money market rate
- Portfolio B*
    - Long put + Long asset

The initial cost of A* is s $C_A + K$ while the cost of B* is $P_A + S_0$. Suppose we go long on portfolio A* and short on portfolio B*. Since the call should never be exercised before T, the only cash flows we must consider is the cash flows from the put option. There are two cash flows possible from the put

- Hold the put until T and receive $\max(0, K - S_T)$
- Exercise the put early
    - Pay K for the stock
    - Close out short position
    - Invest proceeds in the money market

Since this strategy leaves us with a positive cash flow at maturity, the strategy must have a positive cost. Therefore,

$$C_A + K \geq P_A + S_0$$

Suppose now that the underlying asset also pays dividends. When dividends are present, early exercise of the American call may also become optimal. Define the two portfolios

- Portfolio A
    - Long call
- Portfolio B
    - Long put + Investment of $PV(D)$

We argued earlier that the only motivation to exercise the American call early was to receive dividends. However, exercising early would also result in the investor giving up the time value as well as the insurance value of the option. Since portfolio B allow the investor to retain both the time- and insurance value of the option in addition to receiving the dividends early, portfolio B must be worth more than portfolio A

$$C_E + PV(D) \geq C_A$$

Now, add $PV(K)$ on both sides

$$C_E + PV(D) + PV(K) \geq C_A + PV(K)$$

Using the European put-call parity on dividend paying assets give us

$$P_E + S_0 \geq C_A + PV(K)$$

Since American puts are worth more than European puts, we finally get

$$P_A + S_0 \geq C_A + PV(K)$$

We can also find an upper bound for the American put, namely

$$P_A + S_0 \leq C_A + K + PV(D)$$

Put-Call Parity with Dividend Yields

In the section above we have used discrete dividends. Suppose now that the dividends are continuous with a dividend yield of δ. Therefore, if we want one unit of the underlying asset at time T, we need to purchase $e^{-\delta T}$ units of it today.

We derive the modified Put-Call parity by constructing the following portfolios

- Portfolio A
  - Long European call + Investment of $PV(K)$
- Portfolio B
  - Long European put + Long $e^{-\delta T}$ units of the asset

The costs of the portfolios are $C_E + PV(K)$ and $P_E + S_0 e^{-\delta T}$, respectively. Since the portfolios yield the same payoffs, the cost must be the same. So,

$$C_E + PV(K) = P_E + S_0 e^{-\delta T}$$

Generalized Parity

We can use the Put-Call parity for other assets as well. Suppose we have an option to exchange an asset with price S_t for another asset with price Q_t. Now, let $F_{t,T}^P(S)$ and $F_{t,T}^P(Q)$ denote the prepaid forward prices on each asset. The call price and put price with underlying S and strike Q with time to maturity $T - t$ are $C(S_t, Q_t, T - t)$, $P(S_t, Q_t, T - t)$, respectively.

Suppose we want to replicate the payoff of S_t. Then we do the following

Transaction	$t = 0$	$t = T, S_T > Q_t$	$t = T, S_T < Q_T$
Long call	$-C$	$S_T - Q_T$	0
Short put	$+P$	0	$S_T - Q_T$
Long prepaid forward on Q	$-F_{0,T}^P(Q)$	Q_T	Q_T
Total	$P - C + -F_{0,T}^P(Q)$	S_T	S_T

Replication of the asset S will then have an initial cost of $P(S_t, Q_t, T) - C(S_t, Q_t, T) - F_{0,T}^P(Q)$. The initial cost of the asset S is then its time 0 price S_0, which is also the prepaid forward price $F_{0,T}^P(S)$.

We can then claim that in any period, we must have

$$P(S_t, Q_t, T) - C(S_t, Q_t, T) = F_{0,T}^P(S) - F_{0,T}^P(Q)$$

This is the general parity condition.

Option Profits

The option payoff does not take account of the initial cost of entering the position. When an option is purchased for P_0 at time 0, the cost of this position at time T is $P_0 e^{rT}$ because the premium payment could have been invested at the risk-free rate instead. Suppose a call option has a premium of C_0 while a put option has a premium of P_0, then the respective **profits** for the long position in each option at time T is

$$Call\ profit = \max(S_T - K, 0) - C_0 e^{rT}$$

$$Put\ profit = \max(K - S_T, 0) - P_0 e^{rT}$$

For the short position, or the written option, the profits become

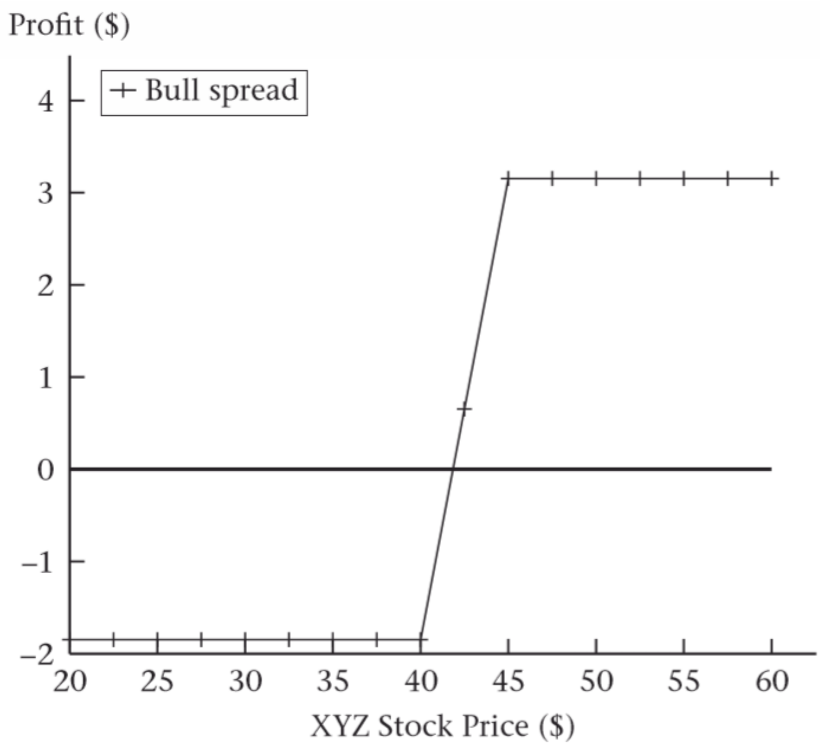
$$Written\ call\ profit = C_0 e^{rT} - \max(S_T - K, 0)$$

$$Written\ put\ profit = P_0 e^{rT} - \max(K - S_T, 0)$$

Directional Option Strategies

It is possible to create option strategies that payoff if the stock moves in a certain direction. This is a **directional** position. We can categorize directional positions as being a **spread** or **collar**. Option spreads are positions that consist of only calls or only puts where some of the options may be written. There are three common spreads: the **bull spread**, the **bear spread**, and the **ratio spread**. The bull spread is performed by entering a long position on a call with strike K_1 and premium C_0^1 and entering a short position on an otherwise identical call with strike $K_2 > K_1$ with premium C_0^2. The profit on this position is

$$Bull\ Spread = \max(S_T - K_1, 0) - \max(S_T - K_2, 0) - C_0^1 e^{rT} + C_0^2 e^{rT}$$



This strategy is useful if the investor believes in an appreciation of the underlying asset.

The opposite of the bull spread is the bear spread. The bear spread is constructed by entering a short position on the low-strike call and entering a long position on the high-strike call. The profit becomes

$$Bear\ Spread = -\max(S_T - K_1, 0) + \max(S_T - K_2, 0) + C_0^1 e^{rT} - C_0^2 e^{rT}$$

The profit diagram of the bear spread will be the exact opposite of the bull spread. This makes it clear that the bear spread strategy is useful if the investor believes that the stock price should remain low.

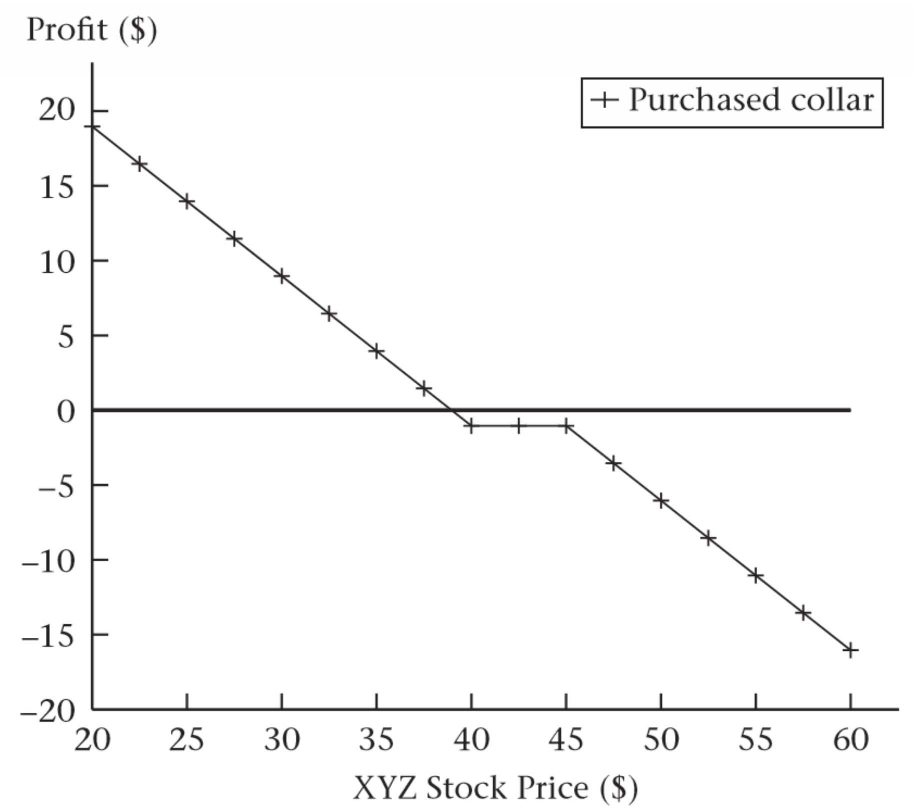
The ratio spread is a position with a long position of m options at strike K_1 and a short position of n options at strike K_2 with $K_1 \neq K_2$. The profit on this strategy will depend on which options are chosen and the ratio m/n.

A second main strategy is a **collar**. A collar consists of a long position in a put option with strike K_1 and a short position of a call option with strike $K_2 > K_1$. The reversed position (short put and long call) is a written collar. The difference $K_2 - K_1$ is called the **collar width**.

The profit on a collar is

$$Collar = \max(K_1 - S_T, 0) - \max(S_T - K_2, 0) - P_0 e^{rT} + C_0 e^{rT}$$

with a profit diagram



Volatility Strategies

We can create option strategies that pay off provided the underlying asset moves in price, regardless of direction. This is a **nondirectional** position. Since these strategies yield payoffs when the asset moves in price, nondirectional strategies are simply volatility strategies. There are three main nondirectional strategies: **straddles**, **strangles**, and **butterfly spreads**.

The straddle is constructed by purchasing a put option and a call option with the same strike price. This is a directional strategy because an increase in the underlying asset will yield a payoff from the call, while a decrease in the underlying asset will give a payoff from the put option. This is essentially a long position on volatility.

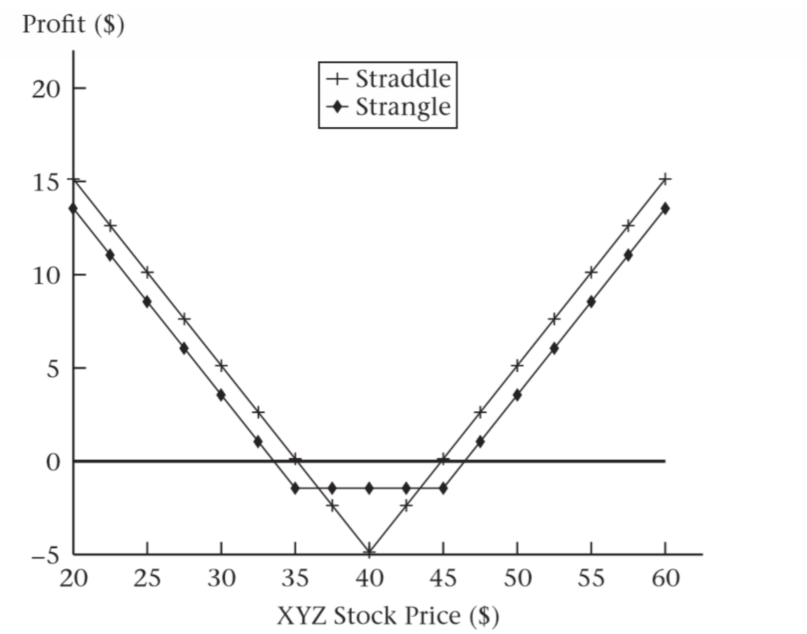
Straddle profits can be expressed as

$$Straddle = \max(S_T - K, 0) + \max(K - S_T, 0) - C_0 e^{rT} - P_0 e^{rT}$$

To avoid the high premiums from the straddle, the investor can instead buy a strangle. The strangle position is entered by purchasing one or both options out-of-the-money. This reduces the premium of the position but will require larger stock price movements to pay off. The profit on the strangle is

$$Strangle = \max(S_T - K_1, 0) + \max(K_2 - S_T, 0) - C_0 e^{rT} - P_0 e^{rT}$$

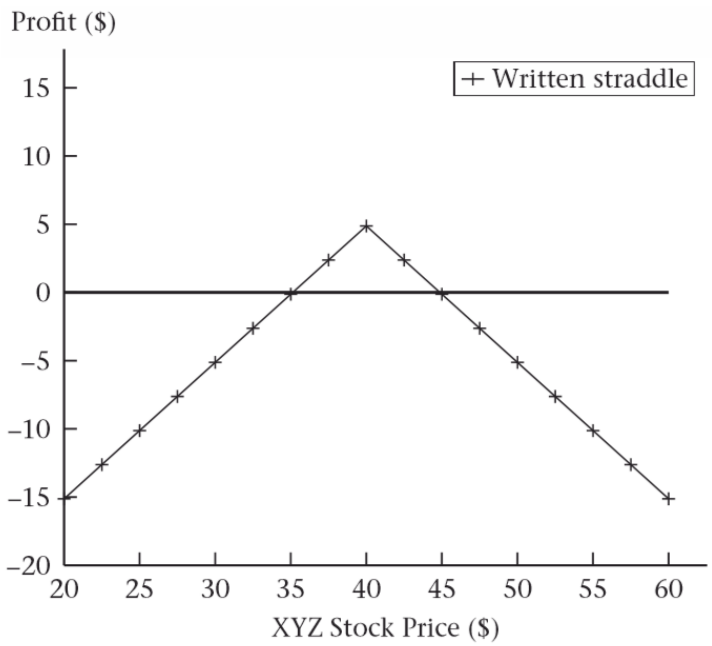
Where $K_1 > S_0$ and/or $S_0 > K_2$. The profit diagram for both straddles and strangles is



The investor can instead go short on volatility by writing a straddle with profits

$$Straddle = -\max(S_T - K, 0) - \max(K - S_T, 0) + C_0 e^{rT} + P_0 e^{rT}$$

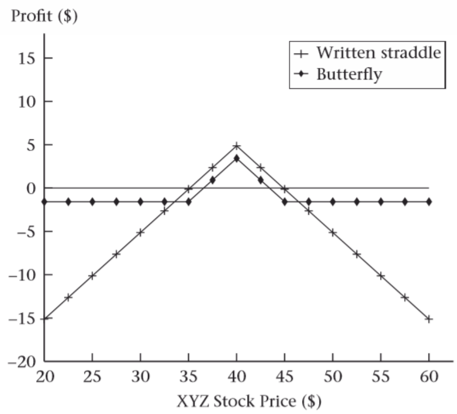
The profit diagram is



The disadvantage of this position is that the upside is limited while the downside is unlimited. To avoid extreme losses, the straddle writer can hedge the position by purchasing out-of-the-money put and call options. The put will hedge against low stock prices while the call will hedge against high stock prices. The combination of a written straddle and insurance is called a **butterfly spread**. Let P_0^O and C_0^O denote the premiums on the out-of-the-money puts and calls, respectively. Then the profit on the butterfly spread is

$$Butterfly\ Spread = -\max(S_T - K, 0) - \max(K - S_T, 0) + C_0 e^{rT} + P_0 e^{rT} - P_0^O e^{rT} - C_0^O e^{rT}$$

The profit diagram is



Chapter 5: Binomial Option Pricing

We will now look at how to determine option prices based on information about the underlying asset. To do this, we will have to posit a model of how the price of the underlying evolves over time. The first model is the **binomial model** which will be discussed in this chapter and the second model is the *Black-Scholes* model which is discussed in chapter 6.

The general binomial model assumes that the stock price can take two values in the next period

$$S_{t+1} = \begin{cases} uS_t & with\ probability\ p \\ dS_t & with\ probability\ 1-p \end{cases}$$

Where $u > d$. The parameters indicate the gross return of the stock from period t to $t+1$. The price changes in the model occur at specified points in time $t = 0, 1, 2, \ldots$ The calendar time between two time points is h years. The parameter h can be very small (days or hours) or very large (years or months).

The volatility of the stock price is related to the ratio u/d. If this ratio is large, the upside return u is much larger than the downside return d. I

In general, these two return parameters are defined by

$$u = e^{(r-\delta)h+\sigma\sqrt{h}}, d = e^{(r-\delta)h-\sigma\sqrt{h}}$$

Before we derive the model, we need the parameter restriction

$$d < e^{(r-\delta)h} < u$$

If $e^{(r-\delta)h} \geq u$, the risk-free asset dominates the stock. Then, the stock can be shorted with the proceeds invested at the risk-free rate. This will result in arbitrage profits. If $e^{(r-\delta)h} \leq d$, the stock dominates risk-free returns so that arbitrage profits can be made by shorting the risk-free asset and investing the proceeds in the risky asset.

Pricing by Replication

Assume the underlying asset pays a dividend yield of δ and that the gross interest rate factor per period is e^{rh}. Then, let $S_h = uS$ and $S_d = dS$, where S is the current stock price. Furthermore, define C_h and C_d as the option payoffs when the underlying takes the values S_h and S_d, respectively.

To find the option price at time 0, we will have to replicate the option payoffs. More specifically, we want to find a portfolio consisting of Δ units of a stock and dollar amount B in borrowing such that the portfolio payoff will create the payoff of the option in both states. Here, $\Delta > 0$ represents a purchase of Δ units while $\Delta < 0$ represents a short sale of Δ units. Similarly, $B > 0$ is to be interpreted as borrowing while $B < 0$ is interpreted as lending.

To replicate the option, the portfolio must pay C_u if the stock moves up and C_d if it moves down. Keeping in mind that a purchase of one stock for S today gives $Se^{\delta h}$ in period h and lending B yield Be^{rh} in period h, we write the system of equations

$$\left(\Delta \cdot uS_0 \cdot e^{\delta h}\right) + Be^{rh} = C_u$$
$$\left(\Delta \cdot dS_0 \cdot e^{\delta h}\right) + Be^{rh} = C_d$$

Which solves for

$$\Delta = \frac{C_u - C_d}{S_0(u-d)} e^{-\delta h}$$

$$B = \frac{uC_d - dC_u}{u-d} e^{-rh}$$

The cost of creating this portfolio must then be

$$\mathrm{C}_0 = \Delta S_0 + B = \left(C_u \cdot \frac{e^{(r-\delta)h} - d}{u-d} + C_d \cdot \frac{u - e^{(r-\delta)h}}{u-d}\right) e^{-rh}$$

Then by the no-arbitrage condition, this must be equivalent to the option price regardless whether it is a put or call. If this relationship does not hold, it is possible to make an arbitrage profit.

- $C_0 > \Delta S_0 + B$
  - Sell the option
  - Purchase Δ shares
  - Borrow B
- $C_0 < \Delta S_0 + B$
  - Purchase the option
  - Sell Δ shares
  - Lend B

The option price can also be stated differently. If we define the risk neutral probability of an upward move of the stock price

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

Then the cost of the replication portfolio $\Delta S_0 + B$ can instead be expressed as

$$\Delta S_0 + B = (p^* C_u + (1 - p^*) C_d) e^{-rh}$$

This makes it clear that the option price looks like a probability weighted discounted cash flow.

Pricing Multi-Period European Options

The binomial model can be extended to multiple periods. The pricing method is however not any different from the one period case conceptually, but the calculation involves multiple steps. To price multi-period European options, we must make use of backward induction.

Suppose we are looking at an n period evolvement of the stock price. This involves n up or down (or both) movements in total. By drawing the binomial tree, we will end up with n terminal prices at the end of the period.

To find the time 0 option price for the n period evolvement of the stock, the backward induction method requires us to first find the $n - 1$ option value for each node in the binomial tree. Then, using the results in the $n - 1$ node, we find the $n - 2$ option values. We then follow this procedure until we arrive at the time 0 option price.

Pricing American Options

When pricing an American option using the binomial model, we must factor in the possibility that early exercise may be optimal. First, we remember that early exercise on a non-dividend paying asset is never optimal on an American call. Therefore, we will only consider American puts, but also calls provided the asset pays dividends.

The pricing is relatively simple. Given a node, early exercise is only optimal if the payoff from the early exercise is greater than the value of the option when it is held for another period. Formally, this decision can be described as

$$Call = \max\left(\underbrace{S_t - K}_{Early\ exercise}, \underbrace{[p^* \cdot C(uS_t, K, t+h) + (1-p^*)C(dS_t, K, t+h)]}_{Value\ of\ call\ if\ held\ another\ period}\right)$$

$$Put = \max\left(\underbrace{K - S_t}_{Early\ exercise}, \underbrace{[p^* P(uS_t, K, t+h) + (1-p^*)P(dS_t, K, t+h)]}_{Value\ of\ put\ if\ held\ another\ period}\right)$$

It is however important to test this condition at each node. After testing whether early exercise is optimal, backward induction is used to arrive at the time 0 price of the option.

Options on Other Assets

The binomial model can easily be extended to price options on assets other than stocks but does however require a slight tweak of the assumptions. Assume that there is no uncertainty about the future stock price. That is, we know today which value S_T will take at time T. No uncertainty must then imply that the stock can only yield a rate of return equal to the risk-free rate. Then, the stock price must equal the forward price

$$F_{t,t+h} = S_t e^{(r-\delta)h}$$

If we now add uncertainty in the forward price rather than the stock price, we can model the stock price movements as follows

$$uS_t = F_{t,t+h} e^{\sigma\sqrt{h}}$$

$$dS_t = F_{t,t+h} e^{-\sigma\sqrt{h}}$$

So, provided we have an expression for the forward price of the asset, we can create a binomial tree of those assets. This tree is called a **forward tree**.

Chapter 6: The Black-Scholes Model

The second common option pricing model, and perhaps the most famous one, is the **Black-Scholes model**. It a continuous time model as prices in the model can change continuously rather than at discrete points in time. The Black-Scholes model is widely popular due to its closed-form solution, that is, an explicit formula for the option price.

Assumptions

- Continuously compounded returns are normally distributed and independent over time
- The volatility of the continuously compounded returns is known and constant
- Future dividends are known
- The risk-free rate is known and constant
- No transaction costs or taxes
- The investor can short sell costless and borrow at the risk-free rate

Pricing Options using Black-Scholes

The Black-Scholes formula for a European call on a stock with dividend yield δ is

$$C(S, K, \sigma, r, T, \delta) = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Where $N(x)$ is the cumulative normal distribution function, which is the probability that a randomly drawn number from a normal distribution with $\mu = 0$ and $\sigma^2 = 1$ will be less than x. We can find the formula for European puts using the Put-Call parity for European options,

$$C(S, K, \sigma, r, T, \delta) + PV(K) = P(S, K, \sigma, r, T, \delta) + S_0 e^{-\delta T}$$

$$P(S, K, \sigma, r, T, \delta) = C(S, K, \sigma, r, T, \delta) + Ke^{-rT} - S_0 e^{-\delta T}$$

$$P(S, K, \sigma, r, T, \delta) = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT} - S_0 e^{-\delta T}$$

Using that $1 - N(x) = N(-x)$, we arrive at

$$P(S, K, \sigma, r, T, \delta) = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

Black-Scholes Pricing on Currency Assets

It is possible to modify the formula such that it is appropriate for pricing options on underlying assets that are not necessarily stocks. From the d_1 variable

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

We can see that

$$\ln\left(\frac{Se^{-\delta T}}{Ke^{-rT}}\right) = -\delta T + \ln S - \ln K + rT$$

Using this, we can see that

$$\ln\left(\frac{Se^{-\delta T}}{Ke^{-rT}}\right) + \frac{1}{2}\sigma^2 T = \ln\left(\frac{S}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T$$

Then, d_1 can be rewritten as

$$d_1 = \frac{\ln\left(\frac{Se^{-\delta T}}{Ke^{-rT}}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

We recognize the terms $Se^{-\delta T}$ and Ke^{-rT} as the prepaid forward prices on the stock and the strike. Using this, we can rewrite the Black-Scholes formula to

$$C(S, K, \sigma, r, T, \delta) = F_{0,T}^{P}(S)N(d_1) - F_{0,T}^{P}(K)N(d_2)$$

$$d_1 = \frac{\ln\left(F_{0,T}^{P}(S)/F_{0,T}^{P}(K)\right) + \left(\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

This provides a general formula for option pricing on other assets in terms of prepaid forward prices. One example on such assets are currency assets. Let x_0 be the spot exchange rate. Recognizing the prepaid forward on the currency as

$$F_{0,T}^{P}(x) = x_0 e^{-r_f T}$$

The Black-Scholes formula for currencies becomes

$$C(S,K,\sigma,r,T,\delta) = x_0 e^{-r_f T} N(d_1) - K e^{-rT}(K) N(d_2)$$

$$d_1 = \frac{\ln(x_0 e^{-r_f T}/K e^{-rT}) + \left(\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Deriving the Option Greeks: A preliminary result

The general Black-Scholes formula with time to maturity $T - t$ is given as

$$C(S,K,\sigma,r,T-t,\delta) = S e^{-\delta(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2)$$

$$P(S,K,\sigma,r,T-t,\delta) = K e^{-r(T-t)} N(-d_2) - S e^{-\delta(T-t)} N(-d_1)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Where $N(\cdot)$ is the cumulative standard normal distribution function. We use the result that for any x,

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

To derive the option Greeks, we first must derive an important preliminary result. We will show that

$$S e^{-\delta(T-t)} N'(d_1) - K e^{-r(T-t)} N'(d_2) = 0$$

To prove this, we must show that

$$S e^{-\delta(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} = K e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2}$$

We proceed by simplifying terms.

$$S e^{-\delta(T-t)} e^{-\frac{1}{2}d_1^2} = K e^{-r(T-t)} e^{-\frac{1}{2}d_2^2}$$

$$\frac{Se^{-\delta(T-t)}}{Ke^{-r(T-t)}} = e^{\frac{1}{2}(d_1^2-d_2^2)}$$

Taking the natural log on each side gives

$$\ln S - \delta(T-t) - \ln K - \big(r(T-t)\big) = \frac{1}{2}(d_1^2 - d_2^2)$$

$$\ln\left(\frac{S}{K}\right) + (r-\delta)(T-t) = \frac{1}{2}(d_1^2 - d_2^2)$$

Using the quadratic formula on the right-hand side

$$\ln\left(\frac{S}{K}\right) + (r-\delta)(T-t) = \frac{1}{2}(d_1 - d_2)(d_1 + d_2)$$

Using the definition of d_1 and d_2, we see that

$$d_1 - d_2 = \sigma\sqrt{T-t}$$

$$d_1 + d_2 = 2d_1 - \sigma\sqrt{T-t}$$

Then,

$$\frac{1}{2}(d_1 - d_2)(d_1 + d_2) = \frac{1}{2}(\sigma\sqrt{T-t})(2d_1 - \sigma\sqrt{T-t})$$

Expanding the terms on the right-hand side gives

$$\frac{1}{2}(d_1 - d_2)(d_1 + d_2) = d_1\sigma\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)$$

Insert for d_1 on the right-hand side

$$\frac{1}{2}(d_1 - d_2)(d_1 + d_2) = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \cdot \sigma\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)$$

Simplifying terms give

$$\frac{1}{2}(d_1 - d_2)(d_1 + d_2) = \ln\left(\frac{S}{K}\right) + (r-\delta)(T-t)$$

Which completes the proof.

Option Greeks: Delta

The option Greeks are expressions for the changes in the option prices when the parameters in the Black-Scholes model are changed slightly.

We begin by deriving the Black-Scholes **delta**. This expression gives the change in the option price when the underlying asset changes with one unit (say, one dollar).

Formally, the call option delta is

$$\Delta_C = \frac{\partial C}{\partial S} = e^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S}$$

Since

$$Se^{-\delta(T-t)}N'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

We write

$$\Delta_C = \frac{\partial C}{\partial S} = e^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} - Se^{-\delta(T-t)}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_2}{\partial S}$$

$$\Delta_C = \frac{\partial C}{\partial S} = e^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)}\frac{\partial N(d_1)}{\partial d_1}\underbrace{\left(\frac{\partial d_1}{\partial S} - \frac{\partial d_2}{\partial S}\right)}_{=0}$$

$$\Delta_C = \frac{\partial C}{\partial S} = e^{-\delta(T-t)}N(d_1) > 0$$

We derive the put option delta through the Put-Call parity. Since

$$C + PV(K) = P + Se^{-\delta(T-t)}$$

$$P = C + Ke^{-r(T-t)} - Se^{-\delta(T-t)}$$

Then

$$\Delta_P = \frac{\partial P}{\partial S} = \frac{\partial C}{\partial S} - e^{-\delta(T-t)}$$

$$\Delta_P = e^{-\delta(T-t)}N(d_1) - e^{-\delta(T-t)} < 0$$

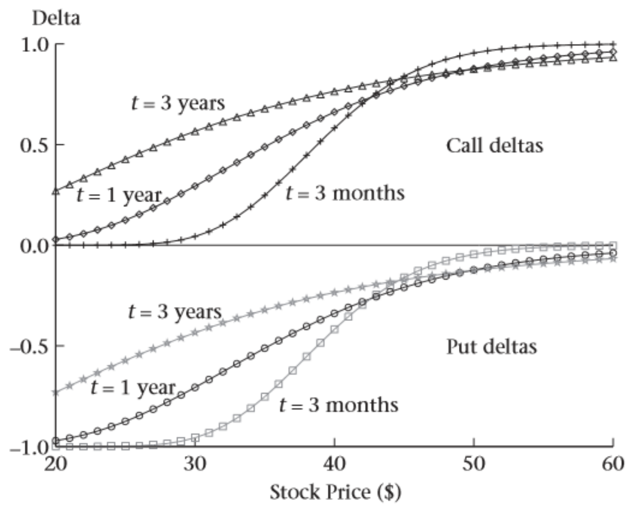
$$\Delta_P = e^{-\delta(T-t)}\big(1 - N(d_1)\big) < 0$$

$$\Delta_P = e^{-\delta(T-t)}N(-d_1) < 0$$

The delta is always positive for a call option which is intuitive as the option payoff increases linearly with the stock price. In fact, $\Delta_C \in [0, 1]$. The delta will approach 1 when the option is deep in-the-money, the reason being that the payoff $(S_T - K) \to S_T$ when S_T becomes large. Therefore, given the strike K, the call option tends to the underlying asset in terms of payoff when the asset price becomes large relative to the strike. Then, $\Delta_C \to 1$. For a put option delta, we have $\Delta_P \in [-1, 0]$.

When the call option is deep out-of-the-money, a dollar increase in the underlying asset will have little effect on the payoff on the call. It will therefore resemble a payoff like that of holding nothing, that is, a zero payoff. In this case, $\Delta_C \to 0$.

The delta will also be different on options with different maturities. When T increases, we would expect the delta to increase when the stock price is low because the probability of a positive payoff will increase. However, if the stock price is high, the probability of landing out-of-the money increases. In this case, the delta would fall.



Another feature of the delta is that it can be interpreted as a share equivalent. This means that the delta will tell us how many units of the underlying asset we need to replicate the option payoff. The synthetic call option is replicated by

$$S\Delta + B = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

Since $\Delta_C = e^{-\delta(T-t)}N(d_1)$, it is clear that

$$B = -Ke^{-rT}N(d_2)$$

So, if we want to replicate the call option, we must purchase $e^{-\delta(T-t)}N(d_1)$ units of the stock and borrow $Ke^{-rT}N(d_2)$ dollars.

Option Greeks: Gamma

The **gamma** is the second derivative of the option price with respect to S, or equivalently the first derivative of the delta. The delta captures the curvature of the price change of the option. Formally,

$$\Gamma_C = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta_C}{\partial S}$$

$$\Gamma_C = e^{-\delta(T-t)} \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial S}$$

$$\Gamma_C = e^{-\delta(T-t)} \frac{\partial N(d_1)}{\partial d_1} \left(\frac{1}{S\sigma\sqrt{T-t}} \right)$$

$$\Gamma_C = \frac{e^{-\delta(T-t)}}{S\sigma\sqrt{T-t}} N'(d_1) > 0$$

Using the put-call parity

$$C + Ke^{-r(T-t)} = P + Se^{-\delta(T-t)}$$

Then

$$\Gamma_P = \Gamma_C > 0$$

Note that the gamma is equal whether it is a put or call. We can also argue intuitively that the gamma will tend to zero when the option is either deep in-the-money or deep out-of-the-money because an additional change in the stock price will have little effect on the value of the option. Therefore, we would expect that the gamma is large when the stock price is somewhere around the strike price.

Option Greeks: Theta

The **theta** measures the price change with respect to the current time t. This is simply

$$\Theta_C = \frac{\partial C}{\partial t}$$

$$\Theta_C = \delta S e^{-\delta(T-t)} N(d_1) + S e^{-\delta(T-t)} \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial t} - rK e^{-r(T-t)} N(d_2) - K e^{-r(T-t)} \frac{\partial N(d_2)}{\partial d_2} \frac{\partial d_2}{\partial t}$$

Since

$$S e^{-\delta(T-t)} N'(d_1) = K e^{-r(T-t)} N'(d_2)$$

We write

$$\Theta_C = \delta S e^{-\delta(T-t)} N(d_1) + S e^{-\delta(T-t)} \frac{\partial N(d_1)}{\partial d_1} \left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} \right) - rK e^{-r(T-t)} N(d_2)$$

Since

$$\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} = -\frac{\sigma}{2\sqrt{T-t}}$$

$$\Theta_C = \delta S e^{-\delta(T-t)} N(d_1) - S e^{-\delta(T-t)} \frac{\partial N(d_1)}{\partial d_1} \frac{\sigma}{2\sqrt{T-t}} - rK e^{-r(T-t)} N(d_2)$$

Using the put call parity, the put option theta becomes

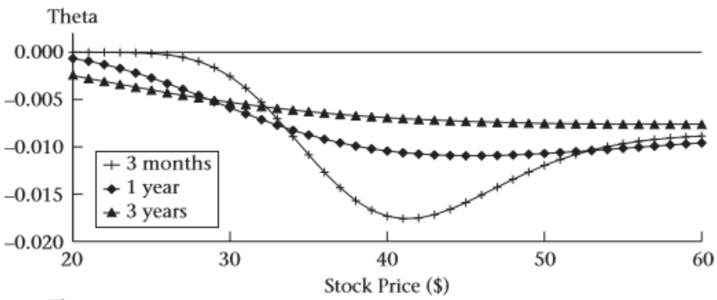
$$C + K e^{-r(T-t)} = P + S e^{-\delta(T-t)}$$

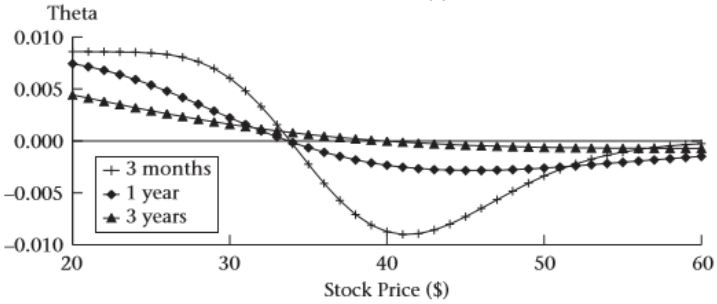
$$\frac{\partial P}{\partial t} = \frac{\partial C}{\partial t} + rK e^{-r(T-t)} + \delta S e^{-\delta(T-t)}$$

$$\Theta_P = \delta S e^{-\delta(T-t)} N(d_1) - S e^{-\delta(T-t)} \frac{\partial N(d_1)}{\partial d_1} \frac{\sigma}{2\sqrt{T-t}} - rK e^{-r(T-t)} N(d_2) + rK e^{-r(T-t)} + \delta S e^{-\delta(T-t)}$$

$$\Theta_P = \Theta_C + rK e^{-r(T-t)} + \delta S e^{-\delta(T-t)}$$

The effect of time is not obvious for options. Generally, one would expect that options become less valuable as time to expiration $T - t$ decreases, that is, a negative theta. This is simply because a small $T - t$ limits the upside potential such that it is unlikely that $S_T - K$ or $K - S_T$ can get significantly large. However, the time decay effect can be positive in some special cases.





Option Greeks: Vega

The **vega** measures the price changes when the volatility of the underlying asset changes by one unit (say, 1 percentage point).

$$V_C = \frac{\partial C}{\partial \sigma}$$

$$V_C = Se^{-\delta(T-t)} \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)} \frac{\partial N(d_2)}{\partial d_2} \frac{\partial d_2}{\partial \sigma}$$

Since

$$Se^{-\delta(T-t)} N'(d_1) = Ke^{-r(T-t)} N'(d_2)$$

$$V_C = Se^{-\delta(T-t)} \left(\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} \right) N'(d_1)$$

We find that

$$\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} = \sqrt{T-t}$$

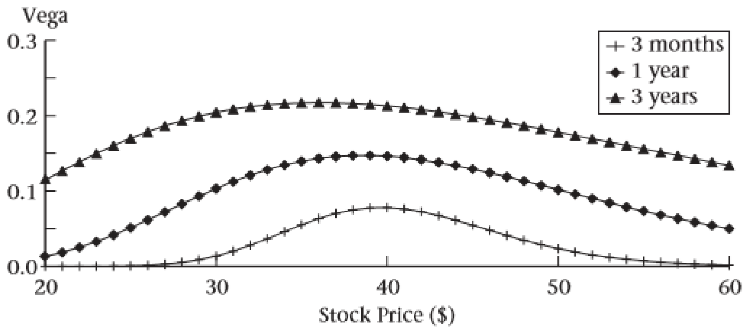
So,

$$V_C = Se^{-\delta(T-t)} N'(d_1) \sqrt{T-t} > 0$$

Volatility does not enter the Put-Call parity besides in the option prices. Then,

$$V_C = V_P$$

Higher volatility in the option price will raise its value as a compensation for taking on risk, so the vega must be positive. We would expect vega to be large when the stock is fluctuating at the strike price because the probability of being out of the money is large. If the option is deep in-the-money or out-of-the-money, the probability of remaining in that state is high, so vega should be small around these stock prices.



Option Greeks: Rho

The **rho** measures the price change of the option when the risk-free rate changes by one unit.

$$\rho_C = \frac{\partial C}{\partial r}$$

$$\rho_C = Se^{-\delta(T-t)} \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial r} + (T-t)Ke^{-r(T-t)} N(d_2) - Ke^{-r(T-t)} \frac{\partial N(d_2)}{\partial d_2} \frac{\partial d_2}{\partial r}$$

Since

$$Se^{-\delta(T-t)} N'(d_1) = Ke^{-r(T-t)} N'(d_2)$$

$$\rho_C = (T-t)Ke^{-r(T-t)} N(d_2) + Se^{-\delta(T-t)} \frac{\partial N(d_1)}{\partial d_1} \underbrace{\left(\frac{\partial d_1}{\partial r} - \frac{\partial d_2}{\partial r}\right)}_{=0}$$

$$\rho_C = (T-t)Ke^{-r(T-t)} N(d_2) > 0$$

Using the Put-Call parity

$$C + Ke^{-r(T-t)} = P + Se^{-\delta(T-t)}$$

The put option rho becomes

$$\frac{\partial P}{\partial r} = \frac{\partial C}{\partial r} + (t - T)Ke^{-r(T-t)}$$

$$\rho_P = (T - t)Ke^{-r(T-t)}N(d_2) + (t - T)Ke^{-r(T-t)}$$

$$\rho_P = (T - t)(N(d_2) - 1)Ke^{-r(T-t)}$$

$$\rho_P = -(T - t)N(-d_2)Ke^{-r(T-t)} < 0$$

We scale the effect to a change of 1 percentage point by dividing the rho by 100.

The rho is positive for a call option. When exercising the call, the investor pays the strike price to receive the underlying. When the interest rate increases, the present value of the strike decreases which increases the present value of the payoff, all else equal. For a put option, the rho is negative. A long position in a put entitles the investor to receive the strike for the asset. A higher interest rate reduces the present value of this payoff and will therefore imply a fall in the price of the put option.

Implied Volatility

One of the inputs in the Black-Scholes option pricing formula is the volatility σ. One alternative is finding an accurate estimate of future volatility. Another alternative is to infer the volatility from the option prices in the options market. An option's **implied volatility** is the volatility that yields the observed option price. Let $\hat{C}$ be the current option price traded in the market. Then,

$$\hat{C} = C(S, K, \sigma_{implied}, r, T - t, \delta)$$

Where $\sigma_{implied}$ is the volatility that returns the current market price. One important detail is that implied volatility does not make a claim on whether the option price is accurate or not, it simply infers the market's assessment of the volatility. Implied volatilities cannot be found directly using the Black-Scholes formula, so you would have to use some sort of numerical procedure.

Market-Making and Delta Hedging

A **market-maker** is an agent who sells to buyers and buy from sellers. The market-maker makes profits by buying financial assets at the bid price and selling them at the ask price. This margin is called the bid-ask spread. Since market-makers are only after profits from the bid-ask spread, they will actively hedge any financial risk associated with the assets they are purchasing. One specific type of hedge is the **delta hedge**. This hedging strategy involves taking an offsetting position in equities to hedge against the risk related to the movement of the underlying assets. A delta hedged position is often not a zero-value position. Therefore, the market-maker must have sufficient capital to cover the position. Since this position eliminates the financial risk, the delta-hedged position should earn the market-maker the risk-free rate.

Suppose that the market-maker has written a call option on one share at $t = 0$ with a price of C_0 with no intentions of hedging against this position. Since $\Delta_0 > 0$, the call option will increase if the stock price increases. If the stock prices increase at time $t = 1$, the option price has increased to C_1. The profit is measured by **marking-to-market** the position, that is, the profit to the market-maker if the position was liquidated today. This one-day profit would be

$$MTM_1 = C_0 - C_1$$

Which would be negative if the stock price had increased. With no hedging, the market-maker would have to accept this loss if the position were liquidated. Alternatively, the market-maker can delta hedge its position by purchasing an off-setting number of shares. As previously discussed, the delta of an option (in addition to borrowing) gives the number of shares that replicate the option payoff.

We can derive an expression for the day-to-day profit of a market-maker that delta hedges its position. Let Δ_i be the option delta on day i, S_i the day i stock price, C_i the day i option price and MV_i the market value of the portfolio on day i.

For every short position on an option, the market-maker hedges the position purchasing $\Delta_i S_i$. The borrowing capacity is then the amount that must be covered by borrowing at the risk-free rate. The borrowing capacity on day i is

$$MV_i = \Delta_i S_i - C_i$$

This makes it clear that if $MV_i > 0$, the market-maker must borrow to cover the position. The day-to-day change in the borrowing capacity is

$$\Delta MV_i = MV_i - MV_{i-1} = \Delta_i S_i - C_i - (\Delta_{i-1} S_{i-1} - C_{i-1})$$

The cost of purchasing additional shares is

$$S_i(\Delta_i - \Delta_{i-1})$$

The interest owed on day i is the interest rate on the previous day's borrowing capacity

$$rMV_{i-1}$$

We can then see that if $MV_{i-1} > 0$, the market-maker must *pay* rMV_{i-1} in interest the next day.

Using this, the net cash flow on day i is the change in market value of the portfolio at day i less the rebalancing at day i and borrowing costs from day $i - 1$

$$NCF_i = \underbrace{MV_i - MV_{i-1}}_{Change\ in\ portfolio} - \underbrace{rMV_{i-1}}_{Borrowing} - \underbrace{S_i(\Delta_i - \Delta_{i-1})}_{Rebalancing}$$

$$NCF_i = \Delta_i S_i - C_i - (\Delta_{i-1} S_{i-1} - C_{i-1}) - rMV_{i-1} - S_i(\Delta_i - \Delta_{i-1})$$

$$NCF_i = \Delta_{i-1} S_i - C_i - (\Delta_{i-1} S_{i-1} - C_{i-1}) - rMV_{i-1}$$

$$NCF_i = \Delta_{i-1}(\Delta S_i) + \Delta C_i - rMV_{i-1}$$

A hedged portfolio that never requires additional cash investment to remain hedged is **self-financing**. This occurs when $NCF_i = 0\ \forall\ i$.

Gamma Hedging

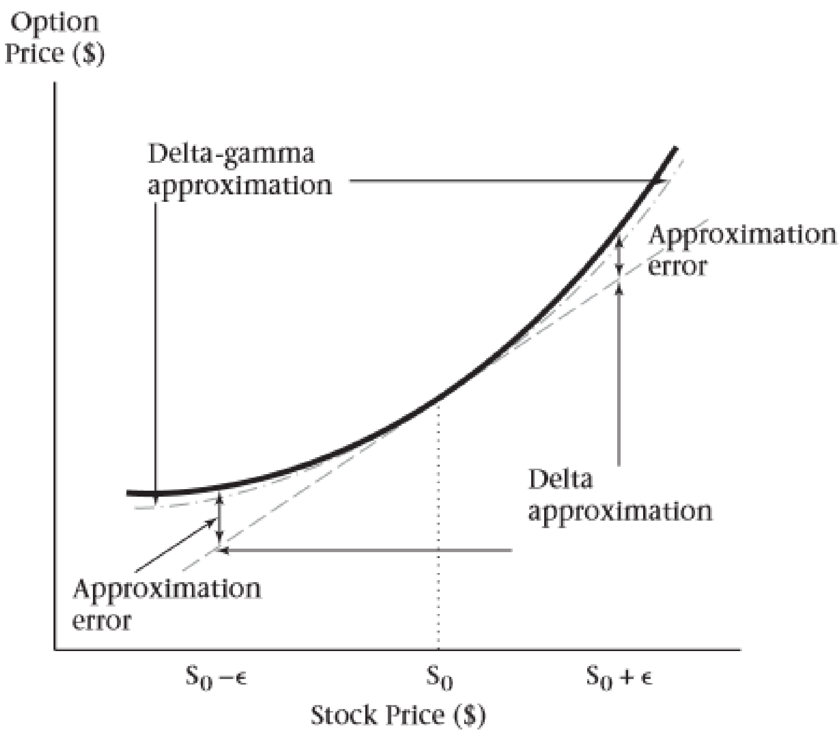
One of the issues with delta hedging in the manner above is that delta changes with the stock price. This is the effect of gamma, as previously discussed. So, when the market-maker hedges by purchasing $\Delta_i S_i$ shares at day i, the position will not be perfectly hedged on day $i+1$. One way to remedy this is to use the **delta-gamma approximation** which accounts for the curvature in the option price as a function of the stock price. Recall that the second order Taylor polynomial is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$

Let $f(x) = C(S_1)$ and $x_0 = S_0$. If $\epsilon = S_1 - S_0$ is small, we can write

$$C(S_1) \approx C(S_0) + \Delta_0 (S_1 - S_0) + \frac{1}{2} \Gamma_0 (S_1 - S_0)^2$$

$$C(S_1) \approx C(S_0) + \Delta_0 \epsilon + \frac{1}{2} \Gamma_0 \epsilon^2$$



Chapter 8: Exotic Options

Exotic options are non-standard options with more complicated payoff structures. We will look at different exotic options.

Currency Options

A **currency** option is an option in which a currency is the underlying asset. While currency options are not exotic, I have added them here for simplicity.

We say that the currency option is **dollar-denominated** if the strike price and premium are quoted in dollars. Similarly, an option with strike and premium quoted in euros is a **euro-denominated** option. A dollar-denominated option, for instance, could be thought of as being based on one unit of a foreign currency.

Currency options have the property that calls can be converted to puts and vice versa. Since purchasing one dollar is equivalent to selling one euro, the right to buy one dollar is equivalent to the right to sell one euro. Therefore, puts in one currency are equivalent to calls in the other currency.

Let x_0 be the USD/EUR exchange rate for instance. Then the call and put prices are related by

$$C_{US}(x_0, K, T) = x_0 K P_{EU}\left(\frac{1}{x_0}, \frac{1}{K}, T\right)$$

Asian Options

An **Asian option** is an option where the payoff depends on the average price over some period. In this case, the average asset price can either be used as the asset price or the strike price. The average is question can either be the ***arithmetic* average**

$$S_a = \frac{1}{N}\sum_{i=0}^{N} S_{t-i}$$

Or the **geometric average**

$$S_g = \left(\prod_{i=0}^{N} S_{t-i}\right)^{1/N}$$

Therefore, when defining an Asian option, the following characteristics must be defined

- Is it a put or call?
- Is the average price used as a strike or as an asset?
- Is the average geometric or arithmetic?

We can then create eight types of Asian options following these characteristics

Asian Option Type	Payoff
Geometric average price call	$\max(0, S_g - K)$
Geometric average price put	$\max(0, K - S_g)$
Geometric average strike call	$\max(0, S_T - S_g)$
Geometric average strike put	$\max(0, S_g - S_T)$
Arithmetic average price call	$\max(0, S_a - K)$
Arithmetic average price put	$\max(0, K - S_a)$
Arithmetic average strike call	$\max(0, S_T - S_a)$
Arithmetic average strike put	$\max(0, S_a - S_T)$

Barrier Options

A **barrier option** has a payoff that depends on whether the price of the underlying reaches a specified level. This specified level is called the ***barrier*.** There are various types of barrier options

Barrier Option Type	Explanation
Knock-out option: Down-and-out	The option becomes worthless if the asset price must fall to reach the barrier
Knock-out option: Up-and-out	The option becomes worthless if the asset price must increase to reach the barrier
Knock-in option: Up-and-in	The option gives a payoff if the asset price must increase to reach the barrier
Knock-in option: Down-and-in	The option gives a payoff if the asset price must fall to reach the barrier

There is a simple relationship between knock-in and knock-out options that are otherwise equal, that is, they have the same barrier H, strike K and maturity T. Consider a portfolio of one knock-out and one knock-in option. If the barrier is never breached before maturity, the knock-out option will give a payoff like a vanilla option while the knock-in option pays nothing. Similarly, if the barrier is breached, the knock-in option pays like a vanilla option while the knock-out pays nothing. Therefore, we can write

$$Knock - out - call + Knock - in - call = Vanilla\ call$$

$$Knock - out - put + Knock - in - put = Vanilla\ put$$

Since option prices can never be negative, it becomes clear that a vanilla option is at least as costly as a barrier option.

Compound Options

A **compound option** is an option to buy an option. Such options complicate the payoff because there are two strike prices and two expiration dates, one for the underlying option and one for the compound option.

Suppose we are at time t_0 and that we have a compound option which at time t_1 will give us the right to pay x for a European call with strike K that expires at T, $T > t_1$. If we exercise the compound call at t_1, then the value of the underlying option we receive is $C(S, K, T - t_1)$. At time T, this option will have the value of a standard call, $\max(0, S_T - K)$. Therefore, the time t_1 value of the compound call is

$$\max\left(C\left(S_{t_1}, K, T - t_1\right) - x, 0\right)$$

Therefore, we will only exercise the compound if the stock price at t_1 is sufficiently large. That is, larger than the compound strike x. Let S^* be the stock price at which $S > S^*$ makes it optimal to exercise the compound call. Then, S^* satisfies

$$C(S^*, K, T - t_1) = x$$

Gap Options

A **gap option** is an option with a payoff that is determined by a trigger price

$$Call\ payoff = \begin{cases} S_T - K_1, S_T > K_2 \\ 0, \qquad else \end{cases}$$

$$Put\ payoff = \begin{cases} K_1 - S_T, S_T > K_2 \\ 0, \qquad else \end{cases}$$

For the vanilla options, the strike and the trigger coincided while the gap option considers separate values of these. An important detail about gap options is that the exercise is forced, which is why we have dropped the max notation for the payoff. Since the exercise is forced, gap options are not really options in the sense that they have to be exercised regardless.

The pricing formula is a slight modification of the Black-Scholes formula. Let K_1 be the strike price and K_2 the trigger price. The pricing formula for a call option is then

$$C(S, K_1, K_2, \sigma, r, T, \delta) = Se^{-\delta T}N(d_1) - K_1e^{-rT}N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{Se^{-\delta T}}{K_2e^{-rT}}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Using the Put-Call parity, we can derive the price for a gap-put.

$$C(S, K_1, K_2, \sigma, r, T, \delta) + Ke^{-rT} = P(S, K_1, K_2, \sigma, r, T, \delta) + Se^{-\delta T}$$

$$P(S, K_1, K_2, \sigma, r, T, \delta) = C(S, K_1, K_2, \sigma, r, T, \delta) + K_1e^{-rT} - Se^{-\delta T}$$

$$P(S, K_1, K_2, \sigma, r, T, \delta) = Se^{-\delta T}N(d_1) - K_1e^{-rT}N(d_2) + K_1e^{-rT} - Se^{-\delta T}$$

$$P(S, K_1, K_2, \sigma, r, T, \delta) = Se^{-\delta T}(N(d_1) - 1) + K_1e^{-rT}\big(1 - N(d_2)\big)$$

$$P(S, K_1, K_2, \sigma, r, T, \delta) = K_1e^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

European Exchange Options

An **exchange option** is an option with the payoff

$$\max\,(0, S_T^1 - S_T^2)$$

where S_T^1, S_T^2 are both risky assets. Therefore, the option will only pay off if the underlying asset S^1 outperforms the benchmark asset S_T^2.

Let S_t^1 and S_t^2 be the market price on the two assets with respective dividend yields δ_1, δ_2 and volatilities σ_1, σ_2. Let ρ denote the correlation between the two asset returns. Then the value of the exchange option is given as

$$V = e^{-\delta_1 T} S^1 N\left(\tilde{d}_1\right) - e^{-\delta_2 T} S^2 N(\tilde{d}_2)$$

$$\tilde{d}_1 = \frac{\left(\ln\left(\frac{S^1}{S^2}\right) + (\delta_2 - \delta_1 + \frac{1}{2}\sigma^2)T\right)}{\sigma\sqrt{T}}$$

$$\tilde{d}_2 = \tilde{d}_1 - \sigma\sqrt{T}$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$$

The exchange option pricing formula collapses to the standard Black-Scholes formula for a call option if $\delta_2 = r$ and $\sigma_2 = 0$, that is, when the benchmark asset is considered risk-free. If we instead set $\delta_1 = r$ and $\sigma_1 = 0$ the formula instead reduces to the Black-Scholes formula for a put option.

Perpetual American Options

Perpetual options are options that never expire. Since the time to expiration is infinity, the exercise problem will look the same in each period. The optimal exercise strategy for perpetual American options is to pick the exercise barrier that maximizes the option price and then exercising the option the first time the stock price reaches the barrier.

Define

$$h_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

$$h_2 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

The value of a perpetual American call option C_∞ with strike K that is exercised when $S \geq H_C$ is

$$C_\infty = (H_C - K)\left(\frac{S}{H_C}\right)^{h_1}$$

Where

$$H_C = K\left(\frac{h_1}{h_1 - 1}\right)$$

Then

$$C_\infty = \left(K\left(\frac{h_1}{h_1-1}\right) - K\right)\left(\frac{S}{K\left(\frac{h_1}{h_1-1}\right)}\right)^{h_1}$$

$$C_\infty = K\left(\frac{h_1}{h_1-1} - 1\right)\left(\frac{1}{\frac{h_1}{h_1-1}}\right)^{h_1}\left(\frac{S}{K}\right)^{h_1}$$

$$C_\infty = K\left(\frac{1}{h_1-1}\right)\left(\frac{S}{K}\right)^{h_1}\left(1 - \frac{1}{h_1}\right)^{h_1}$$

The value of a perpetual American put P_∞ that is exercised when $S \leq H_P$ is

$$P_\infty = (K - H_P)\left(\frac{S}{H_P}\right)^{h_2}$$

Where

$$H_P = K \cdot \frac{h_2}{h_2 - 1}$$

Using this, the option value can be written as

$$P_\infty = \left(K - K \cdot \frac{h_2}{h_2 - 1}\right)\left(\frac{S}{K \cdot \frac{h_2}{h_2 - 1}}\right)^{h_2}$$

$$P_\infty = K\left(1 - \frac{h_2}{h_2 - 1}\right)\left(\frac{S}{K}\right)^{h_2}\left(\frac{h_2 - 1}{h_2}\right)^{h_2}$$

$$P_\infty = -\frac{K}{h_2 - 1}\left(\frac{S}{K}\right)^{h_2}\left(1 - \frac{1}{h_2}\right)^{h_2}$$

Chapter 9: Basic Probability Theory

The Normal Distribution

A random variable $\tilde{x}$ is **normally distributed** if the probability that $\tilde{x}$ takes on a value is described by the **normal density function**

$$\phi(x,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Where μ and σ is the mean and standard deviation of the random variable, respectively. A special case of this function is the **standard normal density function** which occurs when $\mu = 0$ and $\sigma = 1$.

If $\tilde{x}$ is normally distributed, we write this is

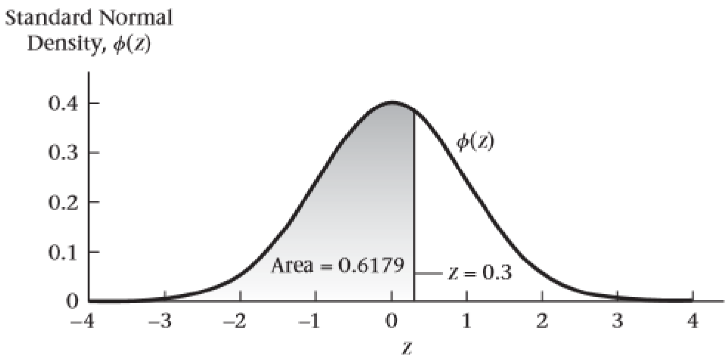
$$\tilde{x} \sim \mathcal{N}(\mu, \sigma^2)$$

We use z to represent random variables that follows a standard normal distribution, and we write this as

$$z \sim \mathcal{N}(0,1)$$

The normal distribution can be used to compute the probability of different events. However, since the density function is continuous (and thus with infinitely many values), the probability that $\tilde{x}$ will take on some value a is 0. To get around this issue, we will instead compute probabilities of ranges. One example is the probability of $\tilde{x}$ being within the interval $[a, b]$ or below a. The probability that $\tilde{x}$ will take a value below a can be found by using the **cumulative normal distribution function**

$$N(a) = P(z < a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x)^2}$$



The symmetry of the normal distribution yields a simple but useful relationship for calculating probabilities

$$N(a) = 1 - N(-a)$$

Suppose now that we have n random variables $x_i, i = 1, \dots, n$ that are jointly distributed with mean μ_i, variance σ_i^2 and covariance σ_{ij}. The weighted sum of these variables has a mean

$$E\left(\sum_{i=1}^{n} w_i x_i\right) = \sum_{i=1}^{n} w_i \mu_i$$

And variance

$$Var\left(\sum_{i=1}^{n} w_i x_i\right) = \sum_{i=1}^{n} \sum_{i=1}^{n} w_i w_j \sigma_{ij}$$

If the random variables are jointly normally distributed, then the sum of the variables are also normal. That is,

$$\sum_{i=1}^{n} w_i x_i \sim \mathcal{N}\left(\sum_{i=1}^{n} w_i \mu_i, \sum_{i=1}^{n} \sum_{i=1}^{n} w_i w_j \sigma_{ij}\right)$$

The final important property of the normal distributed is stated in the **central limit theorem**. The CLT states that the sum of n independent variables, not necessarily normal, approaches the normal distribution as $n \to \infty$ provided the variance of the distribution is finite.

In the context of asset returns, this would that the continuously compounded return is normal provided the daily returns are independent of each other.

The Lognormal Distribution

A random variable y is **lognormally distributed** if $\ln(y)$ is normally distributed. Suppose that x is normally distributed, then y is lognormal if y can be written as

$$\ln(y) = x$$

Or

$$y = e^x$$

This relationship makes a connection between normally distributed returns and lognormality of stock prices. Suppose that the continuously compounded return between $[0, t]$ is normally distributed. Then,

$$R(0,t) = \ln\left(\frac{S_t}{S_0}\right) \sim \mathcal{N}(\mu, \sigma^2)$$

Exponentiating on both sides give

$$S_t = S_0 e^{R(0,t)} \geq 0$$

Which shows that the stock price is lognormally distributed. This makes it clear that a lognormal stock price cannot be negative.

Using the fact that the sum of normal variables is itself normal, we can show that the product of lognormal variables is normal. Let x_1 and x_2 be normal. Then e^{x_1} and e^{x_2} are lognormal. Since

$$e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$$

We see that the product of the lognormal variables must be lognormal since $x_1 + x_2$ is normal.

Generally, if $x \sim \mathcal{N}(\mu, \sigma^2)$ then

$$E(e^x) = e^{\mu + \frac{1}{2}\sigma^2}$$

We can apply this to derive properties for stock prices that are lognormally distributed. Recall that if the continuously compounded return between $[0, t]$ is normally distributed.

Then,

$$R(0,t) = \ln\left(\frac{S_t}{S_0}\right) \sim \mathcal{N}(\mu, \sigma^2)$$

Exponentiating on both sides give

$$\frac{S_T}{S_0} = e^{R(0,t)}$$

Suppose now that we have $t_0 < t_1 < t_2$. Then the stock price at t_1 and t_2 is given as

$$S_{t_1} = S_{t_0} e^{R(t_0,t_1)}$$

$$S_{t_2} = S_{t_1} e^{R(t_1,t_2)}$$

Using the t_1 stock price, we see that the t_2 stock price becomes

$$S_{t_2} = S_{t_0} e^{R(t_0,t_1)+R(t_1,t_2)}$$

Therefore, the continuously compounded return between $[0,t]$ is the sum of returns over each sub-period. If these periods are evenly split with time distance $h = t/n$, we can write

$$R(0,t) = \sum_{i=1}^{n} R\big((i-1)h, ih\big)$$

Suppose that these returns are independent and identically distributed with mean μ and variance σ_h^2, then

$$E[R(0,t)] = n\mu_h$$

$$Var[R(0,t)] = n\sigma_h^2$$

Under these conditions, we see that

$$\ln\left(\frac{S_t}{S_0}\right) \sim \mathcal{N}(n\mu_h, n\sigma_h^2)$$

One common assumption is to assume that

$$E\big(R(0,t)\big) = \left(\alpha - \delta - \frac{1}{2}\sigma^2\right)t$$

$$Var\big(R(0,t)\big) = \sigma^2 t$$

The Conditional Expectations Operator

If S_t is lognormal it is possible to calculate probabilities of future stock prices.

We have

$$S_t = S_0 e^{R(0,t)}$$

Then

$$\ln S_t = \ln S_0 + R(0,t)$$

$$E(\ln S_t) = E(\ln S_0) + E\big(R(0,t)\big)$$

Assuming that

$$R(0,t) \sim \mathcal{N}\left(\left[r - \delta - \frac{1}{2}\sigma^2\right]t, \sigma^2 t\right)$$

$$E(\ln S_t) \sim \mathcal{N}\left(\ln S_0 + \left[r - \delta - \frac{1}{2}\sigma^2\right]t, \sigma^2 t\right)$$

We create the standard normal variable Z by subtracting the mean and dividing by the standard deviation

$$Z = \frac{\ln(S_t) - \ln S_0 - \left[r - \delta - \frac{1}{2}\sigma^2\right]t}{\sigma\sqrt{t}}$$

Since $S_t < K$ implies $\ln(S_t) < \ln(K)$, we know that $P(S_t < K) = P(\ln S_t < \ln K)$. Subtract the mean of $\ln\,(S_t)$ and divide by the standard deviation of $\ln(S_t)$ on both sides inside the probability operator.

Then,

$$P(S_t < K) = P\left(\frac{\ln(S_t) - \ln S_0 - \left[r - \delta - \frac{1}{2}\sigma^2\right]t}{\sigma\sqrt{t}} < \frac{\ln(K) - \ln S_0 - \left[r - \delta - \frac{1}{2}\sigma^2\right]t}{\sigma\sqrt{t}}\right)$$

$$P(S_t < K) = P\left(Z < \frac{\ln(K) - \ln S_0 - \left[r - \delta - \frac{1}{2}\sigma^2\right]t}{\sigma\sqrt{t}}\right)$$

Since $Z \sim \mathcal{N}(0,1)$, we can use the cumulative distribution function to calculate this probability.

The probability is then

$$P(S_t < K) = N(-d_2)$$

And so

$$P(S_t > K) = 1 - P(S_t \leq K)$$

$$P(S_t > K) = N(d_2)$$

Where

$$d_2 = \frac{\ln(S_0) - \ln K + \left[r - \delta - \frac{1}{2}\sigma^2\right]t}{\sigma\sqrt{t}} - \sigma\sqrt{t}$$

We can now use this to calculate the conditional probability. Suppose we want to the expected value of the stock price *conditional* on the stock price being less than K. This expectation is given as

$$E(S_t | S_t < K) = S_0 e^{(r-\delta)t} \cdot \frac{N(-d_1)}{N(-d_2)}$$

Chapter 10: Monte Carlo Valuation

Many derivatives are easy to value as there are valuation formulas such as the Black-Scholes formula. However, for some exotic options such as Asian options, such formulas do not exist. The binomial approach to pricing such derivatives is time consuming and difficult because the options are path dependent. We can use **Monte-Carlo simulations** to simulate future stock prices and discount these to arrive at the price of the option in question.

One assumption we can make is that the assets will earn the risk-free rate of return on average. Using this, we can compute the time 0 value of the option by calculating

$$V[S(0), 0] = e^{-rT} E_0^*[V(S,T), T]$$

Where $V[S(0), 0]$ is the time 0 value of the option and $E_0^*[V(S,T), T]$ is the risk-neutral expected payoff of the option at time T. When pricing options using Monte Carlo, we draw n time T stock prices $S_T^1, \ldots, S_T^n$ randomly and discount the option payoffs back to time 0. Then we take the average of these payoffs.

The time 0 price can be expressed as

$$V(S_0, 0) = \frac{1}{n} e^{-rT} \sum_{i=1}^{n} V\left(S_T^i, T\right)$$

A call option would for instance have $V\left(S_T^i, T\right) = \max\left(S_T^i - K, 0\right)$. When simulating the stock prices, we should expect considerable variation among the individual stock prices from each simulation. We can derive an estimate for this variation in the option price. If there are n trials, the Monte Carlo estimate of the option price is

$$\bar{C}_n = \frac{1}{n} \sum_{i=1}^{n} C\left(\tilde{S}_i\right)$$

Where $C\left(\tilde{S}_i\right)$ is the option price resulting from the randomly drawn stock price $\tilde{S}_i$. If the stock prices are independent and identically distributed, then

$$Var(\bar{C}_n) = Var\left(\frac{1}{n} \sum_{i=1}^{n} C\left(\tilde{S}_i\right)\right)$$

$$Var(\bar{C}_n) = \frac{1}{n^2} \left(\sum_{i=1}^{n} Var\left(C\left(\tilde{S}_i\right)\right)\right)$$

$$Var(\bar{C}_n) = \frac{1}{n^2} (n\sigma_C^2)$$

$$\sigma_n^2 = \frac{1}{n} \sigma_C^2$$

$$\sigma_n = \frac{1}{\sqrt{n}} \sigma_C \xrightarrow{n \to \infty} 0$$

Chapter 11: Credit Risk

Introducing Credit Risk

Credit risk is the risk associated with the probability of the counterparty failing to meet their financial obligations. Failure to make the promised payment is referred to as a **default**.

Suppose that a firm with asset value A_0 issues a zero-coupon bond with face value $\bar{B}$ that matures at time T. Denote B_t as the market value of the bond at time t. At time T, two outcomes are possible

- $A_T > \bar{B}$. The shareholders can repay the bondholders in full. Therefore, $B_T = \bar{B}$.
- $A_T < \bar{B}$. The shareholders are not able to repay the bondholders in full. They can only pay the total assets they own. Therefore, $B_T = A_T$.

Let $g^*(A_T, A_0)$ denote the risk-neutral probability density for the time T asset value conditional on the asset value at time 0, A_0. Pricing the initial value of the debt at $t = 0$ follows a simple concept. The expected payoff at time T is the weighted probability of full repayment and partial repayment (default) discounted back to $t = 0$. In continuous time, this can be expressed as

$$B_0 = e^{-rT}\left[\int_0^{\bar{B}} A_T g^*(A_T, A_0) dA_T + \bar{B}\int_{\bar{B}}^{\infty} g^*(A_T, A_0) dA_T\right]$$

$$B_0 = e^{-rT}\left[E^*(A_T|A_T < \bar{B}) \cdot P^*(A_T < \bar{B}) + \bar{B}\left(1 - P^*(A_T < \bar{B})\right)\right]$$

$$B_0 = e^{-rT}\left[E^*(A_T|Default) \cdot P^*(Default) + \bar{B}\left(1 - P^*(Default)\right)\right]$$

$$B_0 = e^{-rT}[E^*(A_T|Default) \cdot Q + \bar{B}(1 - Q)]$$

Where

$$Q = P^*(Default)$$

We define the **recovery rate** as the amount the bondholders receive relative to what they are owed. When the firm has defaulted, $A_T = B_T$.

Therefore,

$$E^*(Recovery\ rate) = \frac{E^*(A_T = B_T|Default)}{\bar{B}}$$

$$R = \frac{E^*(B_T|Default)}{\bar{B}}$$

The **loss given default** is the amount that that the bondholders do not receive at default relative to what they are owed

$$E^*(Loss\ given\ default) = \frac{\bar{B} - E^*(B_T|Default)}{\bar{B}}$$

$$L = \frac{\bar{B} - E^*(B_T|Default)}{\bar{B}}$$

$$L = 1 - R$$

With the findings above, we can derive an expression for the **credit spread**. The credit spread is the difference in the yield to maturity on a defaultable bond and an otherwise equivalent default-free bond. A default-free bond has

$$P^*(A_T < \bar{B}) = 0$$

In this case, the bond price collapses to

$$B_0^f = \bar{B}e^{-rT}$$

The yield of this bond is the return over $[0, T]$, which is defined as

$$\ln\left(\frac{\bar{B}}{B_0^f}\right) = rT$$

The annualized risk-free yield r is found by dividing the return over length of the period.

Then,

$$r = \frac{1}{T}\ln\left(\frac{\bar{B}}{B_0^f}\right)$$

Similarly, the annualized yield ρ on a defaultable bond is given as

$$\rho = \frac{1}{T}\ln\left(\frac{\bar{B}}{B_0}\right)$$

The credit spread is then

$$\rho - r = \frac{1}{T}\left(\ln\left(\frac{\bar{B}}{B_0}\right) - \ln\left(\frac{\bar{B}}{B_0^f}\right)\right)$$

$$\rho - r = \frac{1}{T}\left(\ln B_0^f - \ln B_0\right)$$

$$\rho - r = \frac{1}{T}\ln\left(\frac{B_0^f}{B_0}\right)$$

$$\rho - r = \frac{1}{T}\ln\left(\frac{\bar{B}e^{-rT}}{e^{-rT}[E^*(A_T|Default)\cdot Q + \bar{B}(1-Q)]}\right)$$

$$\rho - r = \frac{1}{T}\ln\left(\frac{\bar{B}}{[E^*(A_T|Default)\cdot Q + \bar{B}(1-Q)]}\right)$$

$$\rho - r = \frac{1}{T}\ln\left(\frac{\bar{B}}{[\bar{B}(1-LQ + \bar{B}(1-Q)]}\right)$$

$$\rho - r = \frac{1}{T}\ln\left(\frac{1}{[(1-LQ + (1-Q)]}\right)$$

$$\rho - r = \frac{1}{T}\ln\left(\frac{1}{[(P^*(Default) - LQ + (1-Q)]}\right)$$

$$\rho - r = \frac{1}{T}\ln\left(\frac{1}{1-LQ}\right)$$

Using a first order Taylor approximation of the right-hand side yields

$$\rho - r \approx \frac{1}{T}LQ$$

Merton’s Default Model

The **Merton model** is a credit risk model used for determining probabilities of bankruptcy. Using the notation above, bankruptcy implies $\bar{B} > A_T$ with the following payoff

State	$\bar{B} > A_T$	$\bar{B} < A_T$
Shareholder	$\bar{B} - A_T$	0
Debt holder	$\bar{B}$	A_T

Since shareholders are interested in maximizing their value of equity and minimizing the payments to debtholders, the debtholders will receive the smallest payment they are entitled to. Therefore, firm’s balance sheet becomes

Assets	Liabilities
A_T	$B_T = \min(A_T, \bar{B})$
	$E_T = \max(A_T - \bar{B}, 0)$
Total: A_T	Total: $\min(A_T, \bar{B}) + \max(A_T - \bar{B}, 0) = A_T$

The firm’s equity E_T is then seen to resemble a call option on the firm’s assets A_T. The market value of debt at time T can be rewritten as

$$B_T = \min(A_T, \bar{B})$$

$$B_T = \bar{B} + \min(A_T - \bar{B}, 0)$$

Using the max operator property $-\max(a, b) = \min(-a, -b)$, we arrive at

$$B_T = \bar{B} - \max(\bar{B} - A_T, 0)$$

The payoff of the debtholders will then resemble a portfolio of a written put option and a time 0 risk-free loan of $\bar{B}e^{-rT}$ that pays $\bar{B}$ at maturity.

Using this information, we can rewrite the time t balance sheet as

Assets	Liabilities
A_t	$B_t = \bar{B}e^{-r(T-t)} - P(A_t, \bar{B}, T-t)$
-	$E_T = C(A_t, \bar{B}, T-t)$
Total: A_T	Total: $\bar{B}e^{-r(T-t)} - P(A_t, \bar{B}, T-t) + C(A_t, \bar{B}, T-t)$

So, to find the time t value of debt, we can use the Black-Scholes put option formula with underlying A_t and strike $\bar{B}$ to find its value. This formula is

$$B_0 = \bar{B}e^{-r(T-t)} - P(A_t, \bar{B}, T-t)$$

$$B_0 = \bar{B}e^{-r(T-t)} - \bar{B}e^{-r(T-t)}N(-d_2) + A_0 e^{-\delta(T-t)}N(-d_1)$$

$$B_0 = \bar{B}e^{-r(T-t)}\big(N(d_2)\big) + A_0 e^{-\delta(T-t)}N(-d_1)$$

Suppose now that the firm's assets can be represented by the lognormal process

$$A_t = A_0 e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)+\sigma\sqrt{t}Z}$$

Then the probability $P(A_t < \bar{B})$, $t \in [0, T]$ is the risk-neutral default probability

$$P(A_t < \bar{B}) = N(-d_2)$$

$$Q = N(-d_2)$$

$$d_2 = \frac{\ln(A_t) - \ln\bar{B} + \left[\alpha - \delta - \frac{1}{2}\sigma^2\right](T-t)}{\sigma\sqrt{T-t}} - \sigma\sqrt{T-t}$$

Here, we found the default probability using the results from chapter 10. Similarly, the expected recovery rate is

$$R = \frac{E^*(A_T | A_T < \bar{B})}{\bar{B}}$$

$$R = \frac{E^*(A_T | A_T < \bar{B})}{\bar{B}}$$

$$R = A_t e^{(r-\delta)(T-t)} \cdot \frac{N(-d_1)}{\bar{B}N(-d_2)}$$

$$d_1 = d_2 + \sigma\sqrt{T-t}$$

Using that the forward price on the asset is

$$F_{t,T} = A_t e^{(r-\delta)(T-t)}$$

We write

$$R = \frac{F_{t,T}(A)N(-d_1)}{\bar{B}N(-d_2)}$$

Then the loss given default is

$$L = 1 - E^*(Recovery\ rate)$$

$$L = \frac{\bar{B}N(-d_2) + F_{t,T}(A)N(-d_1)}{\bar{B}N(-d_2)}$$

Credit Ratings

Credit ratings provide a measure for the credit risk for bonds. These measures are often provided by third parties that attempt to measure the probability of default on a bond. Examples of such third parties are Standard and Poor's, Moody's, and Fitch. Each rating organization uses a similar rating system. Moody's ratings are designated as Aaa, Aa, A, Baa, Ba, B, Caa, Ca, C where Aaa is a high quality bond with very low default probabilities while bonds that are given the rating C are estimated to have a large default probability.

Over time, we would expect bonds to change ratings depending on their performance. We can calculate the probability of a **ratings transition** using a **transition matrix** which is a matrix of transition probabilities. One example is the following

From/to	AAA	AA	A	BBB	BB	B	CCC/C	D	NR
AAA	87.91	8.08	0.54	0.05	0.08	0.03	0.05	0.00	3.25
AA	0.57	86.48	8.17	0.53	0.06	0.08	0.02	0.02	4.06
A	0.04	1.90	87.29	5.37	0.38	0.17	0.02	0.08	4.75
BBB	0.01	0.13	3.70	84.55	3.98	0.66	0.15	0.25	6.56
BB	0.02	0.04	0.17	5.22	75.75	7.30	0.76	0.95	9.79
B	0.00	0.04	0.14	0.23	5.48	73.23	4.47	4.70	11.71
CCC/C	0.00	0.00	0.19	0.28	0.83	13.00	43.82	27.39	14.48

To give some examples of how the matrix is used

- The one-period probability that a AAA rated bond remains AAA rated is 87.91%
- The one-period probability that an AA rated bond transitions to a BBB rated bond is 0.53%

Suppose that the transition matrix with n possible ratings is constant over time. That is, the probabilities remain unchanged for each period. Let $p(i,t\,;j,t+s)$ denote the probability that a firm with the rating on row i on time t will move to the rating in column j over an s-year time horizon. Then, this transition probability can be expressed as

$$p(i,t\,;j,t+s) = \sum_{k=1}^{n} p(i,t\,;k,t+s-1)\cdot p(k,t+s-1\,;j,t+s)$$

Credit Default Swaps

A **single-name credit default swap (CDS)** is an insurance contract on a bond. The seller of this contract, the **protection seller**, must pay the buyer, **the protection buyer**, when the reference bond experiences a credit event such as a default. The payoff for the buyer of the CDS given default is

$$CDS\ Payoff\ = \bar{B} - B_t$$

The payment convention of a CDS is either an upfront payment at time 0 or an annual premium, or both. To price the CDS, we make use of the fact that a portfolio of a CDS and a risky bond should approximately equal a risk-free bond. Let the upfront premium of the CDS be denoted as CDS_0, then

$$\bar{B}e^{-rT} = CDS_0 + \bar{B}e^{-yT}$$

Tranched Structures

Financial institutions such as banks acquire assets that cannot be sold easily. Examples of such assets are individual mortgages and auto loans. To overcome this problem, institutions pool such assets together and create securities based on portfolios of individual assets. This process is called **securitization**. One example of a security created from securitization is the **asset backed security**, which is roughly defined as a security that generates cash flows based on a pool of other (financial) assets. The asset backed security is an example of a **structure**. The structure defines how the claims on the security are distributed among its investors. When the pool of assets is securitized, the security can be structured such that the cash flows from the individual assets depend on which claim the investor has on the securitized asset. This allows the financial institution to prioritize some investors with regards to the cash flows from the securitized asset. A structure of this kind is referred to as a **tranched structure**.

An example of a tranched structure is the **collateralized debt obligation (CDO)**, which is a structure that repackages the cash flow from a pool of assets. The CDO is created by creating the pool of assets and then issuing tranched financial claims on the securitized asset. The motivation for holding CDOs are many. One reason is to get rid of numerous individual assets. By pooling these assets into a CDO, the institution can sell off the assets. Second, regulations may require an investor to hold an arbitrary amount of investment-grade bonds. Since CDOs are essentially a portfolio of risky assets, a large pool of assets will diversify its risk, effectively creating an investment-grade bond. One common tranched structure is to divide the investors into three tranches. The **senior tranche** consists of the investors that receive the first claim on the cash flows from the bonds. This tranche is the least risky. The **mezzanine tranche** receives the second claim after the investors in the senior tranche has been paid. Whatever is left is paid out to the **equity tranche**. Investors in the equity tranche take on the highest risk of the three types of investors.